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DISCREPANCIES OF IRRATIONAL ROTATIONS WITH ISOLATED LARGE PARTIAL QUOTIENTS

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ABSTRACT. We give some refinements of results of Schoissengeier on the discrepancies of irrational rotations. Using these refinements, we give an explanation for the unusual behavior of the discrepancies of irrational rotations based on some specific numbers having isolated large partial quotients in their continued fraction expansion.

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1. Introduction

For a natural number $a > 1$ it is well-known that the leading digit of a^n is distributed over $\{1, 2, \ldots, 9\}$ non-uniformly for large n and the limit distribution (as $n \to \infty$) of the digit k is equal to $\log_{10}(k+1) - \log_{10} k$ (cf. [14]). In case of some specific numbers a, however, the asymptotic behavior of the chi-square test of goodness of fit of the empirical distribution shows highly unusual aspects. For example, in case of $a = 7$, the values of chi-square tests for the leading digit of 7^n show repetitions of "up" and "down" almost periodically with period about 2,500,000 (cf. [12, 13]). The problem of the leading digit of a^n is closely related to the study of irrational rotations based on $\log_{10} a$ and we will show that the unusual phenomena described above for $a = 7$ are caused by a single large digit in the continued fraction expansion of $1 - \log_{10} 7$. We give discrepancy estimates by using some refinements of formulae given by Schoissengeier (cf. [9, 10]). Using such refinements, we present a mathematical explanation for the quadratic function-like shapes of the graph of values of discrepancies, given in Fig. 1. As it turns out, the period between valleys in Fig. 1 is equal to the denominator

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2,455,069 of 6th-order convergent of continued fraction expansion of $1-\log_{10}7$:

$$1-\log_{10}7=\cfrac{1}{6+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{4813+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cdots}}}}}}}}}$$

Similar phenomena occur for other irrational numbers with isolated large partial quotients, for example, $2-\log_{10}33$ and $2-\log_{10}54$ (see Figs. 2 and 3 at the end of our paper). An explanation of this behavior will be given by the general results in Section 4 connecting the local behavior of $D_N^*(n\alpha)$ and the magnitude of isolated large partial quotients in the continued fraction expansion of α.

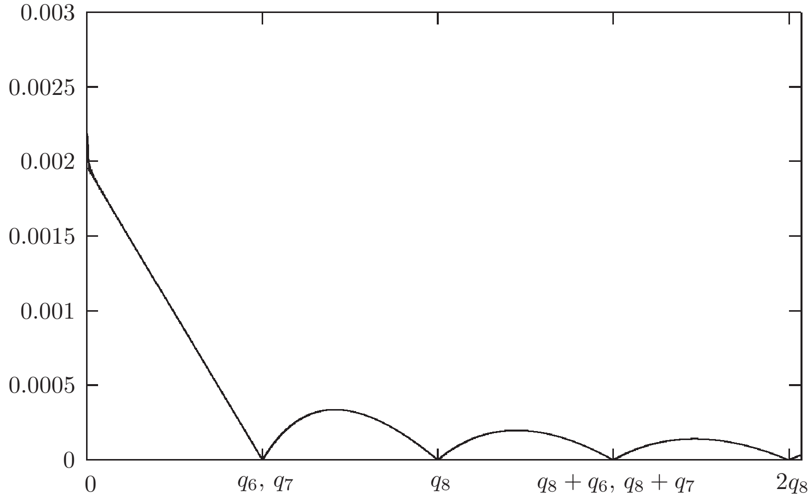

FIGURE 1. $D_N^*(n\alpha)$, $\alpha=1-\log_{10}7$, up to $N=10{,}000{,}000$, every 1,000 points. $q_6=2{,}455{,}069$, $q_7=2{,}455{,}579$, $q_8=4{,}910{,}648$, $q_8+q_6=7{,}365{,}717$, $q_8+q_7=7{,}366{,}227$, $2q_8=9{,}821{,}296$.

In the next section we prepare some basic notions and notations. In Section 3 we recall some results from [9, 10] about discrepancies of irrational rotations and we prove our first main result. In Section 4 we discuss the strange phenomena observed in Fig. 1 in more detail, and we formulate further results. In Section 5 we prove some lemmas which we use later in Section 6. We give proofs for the results of Section 4 in Section 6. Finally, in Section 7 we discuss other examples numerically.

2. Preliminaries

In this section we provide some notions and notations according to Drmota and Tichy [2], Kuipers and Niederreiter [5] and Schoissengeier [9].

For a real number x, let $[x]$ denote the integral part of x, and let $\{x\} = x - [x]$ be the fractional part of x. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be uniformly distributed mod 1 (u.d. mod 1) if for every pair a, b of real numbers with $0 \le a < b \le 1$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[a,b)}(\{x_n\}) = b - a,$$

where $\mathbf{1}_{[a,b)}$ denotes the indicator function of the interval $[a, b)$.

To measure the speed of convergence of a sequence $\omega = (x_n)_{n\in\mathbb{N}}$ to the uniform distribution, we use the following two types of discrepancies D_N and D_N^*:

$$D_N(\omega) = \sup_{0\le a<b\le 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[a,b)}(\{x_n\}) - (b-a) \right|,$$

$$D_N^*(\omega) = \sup_{0<a\le 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[0,a)}(\{x_n\}) - a \right|.$$

It is well-known that a sequence ω is u.d. mod 1 if and only if $\lim_{N\to\infty} D_N(\omega) = 0$, and also $\lim_{N\to\infty} D_N^*(\omega) = 0$ because it holds $D_N^* \le D_N \le 2D_N^*$. In this paper we consider only D_N^*.

Let α be a positive irrational number. We consider an irrational rotation $(n\alpha)_{n\in\mathbb{N}}$ based on α,

$$(n\alpha)_{n\in\mathbb{N}} = \{\{n\alpha\} : n \in \mathbb{N}\},$$

and we denote the discrepancy of $(n\alpha)_{n\in\mathbb{N}}$ by $D_N^*(n\alpha)$. Since $\alpha \mapsto D_N^*(n\alpha)$ is an even and periodic function, that is, $D_N^*(n(1-\alpha)) = D_N^*(n\alpha)$, we restrict α to $0 < \alpha < 1/2$. It is well-known that an irrational rotation $(n\alpha)_{n\in\mathbb{N}}$ is u.d. mod 1, in such a case $\lim_{N\to\infty} D_N^*(n\alpha) = 0$ (cf. [14]).

Let us recall some basic facts about continued fraction expansions. A simple continued fraction expansion is an expression of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, a_3, \ldots]$$

where a_0 is the integer part of α and $a_1, a_2, \ldots$ are positive integers, the so-called partial quotients. Note that $0 < \alpha < 1/2$ implies $a_0 = 0$ and $a_1 > 1$. For $n \geq 0$, the nth-order convergent of the continued fraction α is defined as $r_n = p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$. Here note that r_n's are irreducible fractions, and p_n's and q_n's are determined as follows:

$$\begin{aligned} &p_{-1} = 1,\ p_0 = a_0,\ p_{n+1} = a_{n+1}p_n + p_{n-1}, \\ &q_{-1} = 0,\ q_0 = 1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}. \end{aligned} \tag{1}$$

It is well-known that the convergents of even order form an increasing sequence and that those of odd order form a decreasing sequence. Also, the last convergent, which is equal to α, is greater than any of its even-order convergents and is less than any of its odd-order convergents. Furthermore, the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ satisfy the following inequality:

$$\frac{1}{q_n + q_{n+1}} < |q_n\alpha - p_n| < \frac{1}{q_{n+1}} \tag{2}$$

for all $n \geq 0$.

Next, we give Ostrowski representation of N to base α (cf. [2, 6, 8, 9]). For any given natural number N there is an index m such that $q_m \leq N < q_{m+1}$. By the division algorithm, we have $N = b_m q_m + N_{m-1}$ with $0 \leq N_{m-1} < q_m$. We note that $(a_{m+1} + 1)q_m \geq q_{m+1} > N$, and so $b_m \leq a_{m+1}$. If $m > 0$, we may write $N_{m-1} = b_{m-1}q_{m-1} + N_{m-2}$ with $0 \leq N_{m-2} < q_{m-1}$. Again we find $b_{m-1} \leq a_m$. Continuing in this manner, we arrive at a unique representation for N of the form

$$N = \sum_{j=0}^{m} b_j q_j$$

with $0 \leq b_j \leq a_{j+1}$ for $0 < j < m$, $0 \leq b_0 < a_1$ (since $N_0 = b_0 q_0 < q_1$ and $q_1 = a_1 q_0$) and $0 < b_m \leq a_{m+1}$. Moreover, $b_{j-1} = 0$ if $b_j = a_{j+1}$, since $a_{j+1}q_j + q_{j-1} = q_{j+1}$. Observe that $N_j = \sum_{t=0}^{j} b_t q_t$ for $0 \leq j \leq m$.

Let us define the numbers A_j by

$$A_j = N_{j-1}(\alpha - r_j) + \sum_{t=j}^{m} b_t(q_t\alpha - p_t)$$

for $0 \le j \le m+2$ and $A_{-1} = 0$, where $N_{-1} = 0$ and $N_m = N_{m+1} = N$. We note that the numbers A_j depend on N and α only. The determination of the sign of A_j would be important for estimate of $D_N^*(n\alpha)$. The following lemma states the properties of A_j that we use later.

Lemma 2.1. ([2, Lemma 1.62], [9, Section 3, Proposition 1])

(i) *If* $b_j \neq 0$, *then* $(-1)^j A_j > 0$ *for* $0 \le j \le m$.

(ii) *For* $0 \le j \le m$

$$-|q_j\alpha - p_j| < (-1)^j \sum_{t=j}^{m} b_t(q_t\alpha - p_t) < |q_{j-1}\alpha - p_{j-1}|.$$

(iii) *For* $0 \le j \le m$

$$-\frac{1}{q_{j+1}} < (-1)^j A_j < \frac{1}{q_j}.$$

Lemma 2.1 (iii) implies that $-1 < A_0 < 1$, since $q_1 \ge q_0 = 1$. The numbers A_j also have the property that

$$(-1)^j b_j = a_{j+1} q_j A_j - q_{j+1} A_{j+1} + q_{j-1} A_{j-1} \tag{3}$$

for $0 \le j \le m$ (cf. [10, p.55]). Let $P = \{j : A_j > 0, 0 \le j \le m\}$. In case $j \notin P$ is even, or in case $j \in P$ is odd, Lemma 2.1 (i) shows that $b_j = 0$. In such cases from equation (3) we have

$$a_{j+1} q_j A_j = q_{j+1} A_{j+1} - q_{j-1} A_{j-1} \tag{4}$$

for $0 \le j \le m$.

3. Discrepancies of irrational rotations

In this section we give some refinements of results of Schoissengeier [9, 10] on $ND_N^*(n\alpha)$. Let $i_N = \min\{j \ge 0 : b_j \neq 0\}$ and let

$$s = \min\{j : j \,\text{odd}, A_j > 0, A_{j+2} > 0, b_{j+1} < a_{j+2}, 1 \le j \le m\},$$
$$t = \min\{j : j \,\text{odd}, A_{j-1} < 0 < A_{j+1}, A_{j+2} > 0, b_{j+1} < a_{j+2} - 1, 1 \le j \le m\},$$

where $\min \emptyset = \infty$. We define

$$u = \begin{cases} 0, & \text{if } i_N \text{ is even and } (b_0 < a_1 - 1 \text{ or } A_1 < 0), \\ \min(s,t), & \text{otherwise,} \end{cases}$$

and furthermore sets S_1, S_2, S_3:

$$S_1 = \{j : j \text{ even},\ A_{j+1} < 0 < A_{j-1},\ 0 \le j \le m\},$$
$$S_2 = \{j : j \text{ even},\ A_{j-1} \le 0 < A_{j+1},\ 0 \le j \le m\},$$
$$S_3 = \{j : j \text{ even},\ A_j < 0,\ 0 \le j \le m\}.$$

Using these notations, Schoissengeier [9] gives the following strict formula for $ND_N^*(n\alpha)$:

Theorem A. [9, Section 8, Theorem 1] *Let α be an irrational number, $0 < \alpha < 1/2$. Then*

$$\begin{aligned} ND_N^*(n\alpha) = \sum_{\substack{j=u \\ j \text{ even}}}^{m} b_j(1 - q_jA_j) + \sum_{\substack{j=u \\ j\in S_1}}^{m} q_jA_j - \sum_{\substack{j=u \\ j\in S_2}}^{m} q_jA_j - \sum_{\substack{j=u \\ j\in S_3}}^{m} a_{j+1}q_jA_j \\ + (\delta_{u,0} - 1)q_uA_u + \max\left(0, A_0 - \sum_{j=0}^{m} b_j((-1)^j - q_jA_j)\right). \end{aligned} \tag{5}$$

This result is the very exact formula for discrepancies of irrational rotations. This is, however, not suitable for the estimation of the asymptotic behavior of $D_N^*(n\alpha)$ because the definitions of s, t and u are quite complicated.

In [10], Schoissengeier gives the following formula for $ND_N^*(n\alpha)$ in the proof of Corollary 1 of Theorem 1. His formula is suitable for our proof of Theorem 4.1 (will appear later) and we refer his formula as a separate theorem for convenient reference.

Theorem B. [10, p.56] *Let α be an irrational number, $0 < \alpha < 1/2$. Then*

$$\begin{aligned} ND_N^*(n\alpha) = \sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) + \sum_{\substack{j \text{ odd} \\ j\in P}} a_{j+1}q_jA_j - \sum_{\substack{j \text{ even} \\ j\notin P}} a_{j+1}q_jA_j \\ + \max\left(0, A_0 - \sum_{j=0}^{m} b_j((-1)^j - q_jA_j)\right) + \varepsilon, \end{aligned} \tag{6}$$

where the error term ε satisfies $|\varepsilon| \le 1$, and $P = \{j : A_j > 0,\ 0 \le j \le m\}$.

Note, here, that sets S_1 and S_2 are determined by signs of A_{j-1}'s and A_{j+1}'s, while the set P is determined only by the signs of A_j's. Thus Theorem B makes us to consider only the signs of A_j's. It is, however, difficult to determine the signs of A_j's for general N's. Therefore we want to modify Theorem B so as to do without P in summations, and we obtain the following result:

Theorem 3.1. *Let α be an irrational number, $0 < \alpha < 1/2$. Then*

$$\begin{aligned} &ND_N^*(n\alpha) \\ &= \max\left(\sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}),\ \sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1 + q_{2j+1}A_{2j+1}) + A_0\right) + c, \end{aligned} \tag{7}$$

where the error term c satisfies $-1 \le c < [(m+1)/2] + 1$.

Proof. As in the right-hand side of Theorem B, (6), the first sum combine with the maximum term to form the maximum term of the right-hand side of (7). Hence we prove this theorem by showing that

$$0 \le \sum_{\substack{j \text{ odd} \\ j \in P}} a_{j+1}q_jA_j - \sum_{\substack{j \text{ even} \\ j \notin P}} a_{j+1}q_jA_j < \left[\frac{m+1}{2}\right]. \tag{8}$$

If $j \notin P$ is even, then $j-1 \notin P$ and $j+1 \notin P$. If $j \in P$ is odd, then $j-1 \in P$ and $j+1 \in P$ (cf. [10, p.55]). Moreover, since $b_m \neq 0$, we have $(-1)^m A_m > 0$ by Lemma 2.1 (i). From the above it follows that

$$\begin{aligned} 0 \le \sharp\{j : j \notin P \text{ is even},\ 0 \le j \le m\}& \\ + \sharp\{j : j \in P \text{ is odd},\ 1 \le j \le m\} &\le [(m+1)/2], \end{aligned} \tag{9}$$

where $\sharp(A)$ denotes the number of elements of a set A.

Now we show that $0 < -a_{j+1}q_jA_j < 1$ under the condition $j \notin P$ is even and that $0 < a_{j+1}q_jA_j < 1$ under the condition $j \in P$ is odd. Let us assume first that $j \notin P$ is even. Then it follows from (for even j) $\alpha - r_j > 0$ that

$$\sum_{t=j}^{m} b_t(q_t\alpha - p_t) < A_j < 0.$$

As for the above left-hand side, it is greater than $-|q_j\alpha - p_j|$ by Lemma 2.1 (ii). Hence we have $0 < -a_{j+1}q_jA_j < a_{j+1}q_j|q_j\alpha - p_j|$. This, together with the fact $a_{j+1}q_j|q_j\alpha - p_j| < 1$ (cf. (1) and (2)), implies

$$0 < -a_{j+1}q_jA_j < 1.$$

By a similar argument, we have $0 < a_{j+1}q_jA_j < 1$ under the condition $j \in P$ is odd. From these two estimates and (9), we obtain (8). This completes the proof. $\square$

Equation (7) of Theorem 3.1 seems less strict than those of Theorem A or of Theorem B, but it is simpler because it does not include the set P, and its usefulness will be proved in later discussions on "quadratic-function" like behaviors of $ND_N^*(n\alpha)$. As a by-product of Theorem 3.1, we have the following formula:

Corollary 3.2. *Let α be an irrational number, $0 < \alpha < 1/2$. Then*

$$ND_N^*(n\alpha) = \max\left(\sum_{j=0}^{[m/2]} b_{2j}\left(1 - \frac{b_{2j}}{a_{2j+1}}\right), \sum_{j=0}^{[(m-1)/2]} b_{2j+1}\left(1 - \frac{b_{2j+1}}{a_{2j+2}}\right)\right) + c',$$

where $-2[m/2] - 4 < c' < 3[m/2] + [(m+1)/2] + 5$.

By using Lemma 5.2, we prove this corollary later in Section 5. Schoissengeier [9] gives the following simpler formula:

$$\begin{aligned} &ND_N^*(n\alpha) \\ &= \max\left(\sum_{j=0}^{[m/2]} b_{2j}\left(1 - \frac{b_{2j}}{a_{2j+1}}\right), \sum_{j=0}^{[(m-1)/2]} b_{2j+1}\left(1 - \frac{b_{2j+1}}{a_{2j+2}}\right)\right) + \mathcal{O}(m), \end{aligned}$$

where $\mathcal{O}(m)$ is the Landau's symbol. This formula is considerably simpler than (5) of Theorem A. The error term of this formula, however, is described in the form of Landau's symbol and its magnitude is unclear. Our Corollary 3.2 gives the strict estimate of the error term.

4. Main results

In this section we first give some upper estimates for $ND_N^*(n\alpha)$ with respect to an irrational number α, $0 < \alpha < 1/2$, generally. In the latter part of this section, we give some estimates for $ND_N^*(n\alpha)$ with respect to some specific irrational numbers, each of which has an isolated large partial quotient in its continued fraction expansion.

4.1. Upper estimates of valleys

We first give upper estimates for $D_N^*(n\alpha)$ with respect to some specific values of N. Let α be an irrational number, $0 < \alpha < 1/2$. For the simplest $N = b_m q_m$, with $1 \le b_m \le a_{m+1}$, it holds clearly that

$$A_0 = A_1 = \cdots = A_m = b_m q_m(\alpha - r_m)$$

because $b_0 = b_1 = \cdots = b_{m-1} = 0$. Therefore we have $A_j > 0$ for $0 \le j \le m$, if and only if m is even. Thus we can easily see that $S_1 = \{m\}$, $S_2 = \{0\}$ and

$S_3 = \emptyset$ if m is even and that $S_1 = S_2 = \emptyset$ and $S_3 = \{j : j \text{ even},\ 0 \le j \le m\}$ if m is odd. Also, for odd m, we have

$$A_0 - \sum_{j=0}^{m} b_j((-1)^j - q_j A_j) > 0.$$

By using these properties and Theorem A, Schoissengeier [9] gives the following corollary:

Corollary C. [9, Section 9, Corollary 1] *Let α be an irrational number, $0 < \alpha < 1/2$. Then for $N = b_m q_m$, $1 \le b_m \le a_{m+1}$,*

$$ND_N^*(n\alpha) = b_m - (b_m - 1)b_m q_m |q_m\alpha - p_m| - b_m|q_m\alpha - p_m|. \tag{10}$$

This corollary works well, in case $N = b_m q_m$, with being combined with Theorem A. It is difficult, however, to apply Corollary C and Theorem A to the following cases:

$$N = b_m q_m + b_{m-2} q_{m-2} \quad \text{for } 1 \le b_m \le a_{m+1}, 1 \le b_{m-2} \le a_{m-1},$$
$$N = b_m q_m + b_{m-1} q_{m-1} \quad \text{for } 1 \le b_m < a_{m+1}, 1 \le b_{m-1} \le a_m$$

because the sets S_1, S_2, S_3 and the sign of the term $A_0 - \sum_{j=0}^{m} b_j((-1)^j - q_j A_j)$ in Theorem A, (5), depend not only on m, but also on α, b_m, b_{m-1} and b_{m-2}. Thus we need to derive some formulae of $ND_N^*(n\alpha)$ for N, satisfying the above conditions. By using Theorem B we obtain the following result:

Theorem 4.1. *Let α be an irrational number, $0 < \alpha < 1/2$. Put $K_j = (-1)^j q_j A_j$.*

(i) *For $N = b_m q_m + b_{m-2} q_{m-2}$, $1 \le b_m \le a_{m+1}$, $1 \le b_{m-2} \le a_{m-1}$,*

$$\begin{aligned} ND_N^*(n\alpha) = {} & \max(b_m(1 - K_m) + b_{m-2}(1 - K_{m-2}) - |A_0|, 0) \\ & + \begin{cases} K_{m-2} + \varepsilon, & \text{if } (-1)^{m-1}A_{m-1} > 0, \\ K_m + \varepsilon, & \text{if } (-1)^{m-1}A_{m-1} < 0. \end{cases} \end{aligned} \tag{11}$$

(ii) *For $N = b_m q_m + b_{m-1} q_{m-1}$, $1 \le b_m < a_{m+1}$, $1 \le b_{m-1} \le a_m$,*

$$\begin{aligned} ND_N^*(n\alpha) = {} & \max(b_m(1 - K_m), b_{m-1}(1 - K_{m-1}) - |A_0|) \\ & + K_{m-1} + \varepsilon. \end{aligned} \tag{12}$$

In (11) *and* (12)*, the error term ε's satisfy $|\varepsilon| \le 1$.*

From Corollary C and Theorem 4.1, we can derive the following upper estimates for $D_N^*(n\alpha)$ for some specific N's.

Theorem 4.2. *Let α be an irrational number, $0 < \alpha < 1/2$. Then*

(i)

$$D_N^*(n\alpha) < \frac{1}{q_m} \quad \textit{for } N = b_m q_m,\ 1 \le b_m \le a_{m+1}.$$

(ii)

$$D_N^*(n\alpha) < \frac{b_m + b_{m-2} + 1}{b_m q_m + b_{m-2} q_{m-2}}$$
$$\textit{for } N = b_m q_m + b_{m-2} q_{m-2},\ 1 \le b_m \le a_{m+1},\ 1 \le b_{m-2} \le a_{m-1}.$$

(iii)

$$D_N^*(n\alpha) < \frac{\max(b_m + 1, b_{m-1}) + 1}{b_m q_m + b_{m-1} q_{m-1}}$$
$$\textit{for } N = b_m q_m + b_{m-1} q_{m-1},\ 1 \le b_m < a_{m+1},\ 1 \le b_{m-1} \le a_m.$$

The shapes of graphs of discrepancies are slightly different between cases where α equals $1 - \log_{10} 7$, $2 - \log_{10} 33$, or $2 - \log_{10} 54$, and so on. First we consider mainly the case where $\alpha = 1 - \log_{10} 7$. α has an isolated large partial quotient $a_6 = 4813$. This isolated large partial quotient a_6 plays very important roles in our arguments. We denote the suffix 6 of a_6 by η, that is, $\eta = 6$. Thus, $q_\eta = q_6 = 2455069$, $q_{\eta+1} = q_7 = 2455579$ and $q_{\eta+2} = q_8 = 4910648$ (cf. Table 1). This η may vary, of course, for different α.

Theorem 4.2 gives very good estimates for "valleys" in Fig. 1. Theorem 4.2 (i) shows that the first valley in Fig. 1, $N = q_\eta$ or $q_{\eta+1}$, is very "deep", that is, $D_N^*(n\alpha)$ is very small, $D_N^*(n\alpha) < 1/q_\eta$ or $D_N^*(n\alpha) < 1/q_{\eta+1}$.

Theorem 4.2 (i) also shows that the second valley, $N = q_{\eta+2}$, and the fourth valley, $N = 2q_{\eta+2}$, are also very "deep". In case $N = q_{\eta+2} + q_\eta$, Theorem 4.2 (ii) shows that $D_N^*(n\alpha) < 3/(q_{\eta+2} + q_\eta)$. In case $N = q_{\eta+2} + q_{\eta+1}$, Theorem 4.2 (iii) shows that $D_N^*(n\alpha) < 3/(q_{\eta+2} + q_{\eta+1})$. These estimates show that the third valley is also very "deep".

Note that the first and the third valleys are a little "wider" than the second and the fourth valleys. They have "width" 510. In case of $\alpha = 1 - \log_{10} 7$, we can obtain the upper estimates for the "bottom" of the first and the third valleys. For other α, which has an isolated large partial quotient, we can obtain similar upper estimates for wider valleys. Let us now introduce the following notation:

$$M(N) = \max\left(\sum_{\substack{j=0 \\ j \ne (\eta-1)/2}}^{[m/2]} a_{2j+1},\ \sum_{\substack{j=1 \\ j \ne \eta/2}}^{[(m+1)/2]} a_{2j} \right).$$

By using this $M(N)$, we obtain the following upper estimate for $D_N^*(n\alpha)$ in wider valleys.

Theorem 4.3. *Let α be an irrational number, $0 < \alpha < 1/2$. If*

$$M(N) \geq \left[\frac{m+1}{2}\right] + 2, \tag{13}$$

then we have

$$D_N^*(n\alpha) < \frac{2M(N)}{N} \tag{14}$$

for $N \in I$, where I denotes wider valley intervals, for example, in case of $\alpha = 1 - \log_{10} 7$ or $2 - \log_{10} 33$, $I = [q_\eta, q_{\eta+1})$ or $[q_{\eta+2} + q_\eta, q_{\eta+2} + q_{\eta+1})$. In case of $\alpha = 2 - \log_{10} 54$, $I = [2q_\eta, q_{\eta+1})$ or $[q_{\eta+2} + 2q_\eta, q_{\eta+2} + q_{\eta+1})$, and so on.

Remark 1. In case of $\alpha = 2 - \log_{10} 33$, we can obtain the upper estimate (14) for the "bottom" of the first and the third valleys in Fig. 2, for $\eta = 3$. Also, in case of $\alpha = 2 - \log_{10} 54$, we can obtain the upper estimate (14) for the "bottom" of the second and the fifth valleys in Fig. 3, for $\eta = 7$.

4.2. Estimates for three large hills

In this subsection, we give some estimates for $ND_N^*(n\alpha)$ for "hills", observed in Fig. 1.

Recall, first, that α has very characteristic continued fraction expansion, given in Section 1:

$$1 - \log_{10} 7 = [0; 6, 2, 5, 6, 1, 4813, 1, 1, 2, 2, 2, 1, 1, 1, 6, 5, 1, 83, 7, 2, \ldots].$$

Note that α has a rather large 6th partial quotient, 4813. The nth-order convergents of α are given in Table 1.

Table 1. $\alpha = 1 - \log_{10} 7$ $(0 \leq n \leq 9)$

n	0	1	2	3	4	5	6	7	8	9
a_n	0	6	2	5	6	1	4813	1	1	2
p_n	0	1	2	11	68	79	380295	380374	760669	1901712
q_n	1	6	13	71	439	510	2455069	2455579	4910648	12276875

Let N be a natural number with Ostrowski representation $N = \sum_{j=0}^{m} b_j q_j$ to base α, $\alpha = 1 - \log_{10} 7$. Then the coefficients b_j of N take values, as follows:

$$0 \leq b_0 < 6,\ \ 0 \leq b_1 \leq 2,\ \ 0 \leq b_2 \leq 5,\ \ 0 \leq b_3 \leq 6,\ \ 0 \leq b_4 \leq 1,$$
$$0 \leq b_5 \leq 4813,\ \ 0 \leq b_6 \leq 1,\ \ 0 \leq b_7 \leq 1,\ \ 0 \leq b_8 \leq 2,\ \ \ldots.$$

Note, here, that only the coefficient b_5 can vary very widely. By using this distinctive features of $\alpha = 1 - \log_{10} 7$, we can obtain very strict estimates for $ND_N^*(n\alpha)$.

For other α, which has an isolated large partial quotient, we can show similar estimates. First, we give the following estimate.

Theorem 4.4. *Let α be an irrational number, $0 < \alpha < 1/2$. Assume that α has an isolated large partial quotient a_η satisfying*

$$a_\eta > 12M(N). \tag{15}$$

If

$$M(N) \geq 5, \tag{16}$$

then we have

$$-M(N) < ND_N^*(n\alpha) - b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) < 3M(N) \tag{17}$$

for $N \in I$, where I denotes hill intervals, for example, in case of $\alpha = 1 - \log_{10} 7$ or $2 - \log_{10} 33$, $I = [q_{\eta+1}, q_{\eta+2})$, $[q_{\eta+2}, q_{\eta+2} + q_\eta)$ or $[q_{\eta+2} + q_{\eta+1}, 2q_{\eta+2})$. In case of $\alpha = 2 - \log_{10} 54$, $I = [q_\eta, 2q_\eta)$, $[q_{\eta+1}, q_{\eta+2})$, $[q_{\eta+2}, q_{\eta+2} + q_\eta)$, $[q_{\eta+2} + q_\eta, q_{\eta+2} + 2q_\eta)$ or $[q_{\eta+2} + q_{\eta+1}, 2q_{\eta+2})$, and so on.

Remark 2. This theorem gives numerical explanations for the unusual aspects, shown in Fig. 1. Let $f(x)$ be the quadratic function: $f(x) = x(1-x)$, $0 \leq x \leq 1$. It is well-known that $f(x)$ equals 0 at $x = 0$ and 1, and equals $1/4$ at $x = 1/2$. Then, the term $b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right)$ in Theorem 4.4 is described as follows:

$$b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) = a_\eta f(x),$$

where $x = b_{\eta-1}/a_\eta$, $0 \leq b_{\eta-1} \leq a_\eta$. Thus the estimate in Theorem 4.4 can be rewritten as follows:

$$-M(N) < ND_N^*(n\alpha) - a_\eta f(x') < 3M(N) \quad \text{for } x' = \frac{N'}{a_\eta q_{\eta-1}},$$

where $N' = N - \sum_{j=0, j\neq\eta-1}^{m} b_j q_j$ if $N \in I$. This gives an explanation of "quadratic-function" like behavior of $ND_N^*(n\alpha)$, $\alpha = 1 - \log_{10} 7$, and it is easily seen that the "peak" of each "hill" is estimated from below by

$$a_\eta f\left(\frac{1}{2}\right) - M(N).$$

We now consider the condition (15),

$$a_\eta > 12M(N).$$

For $\alpha = 1 - \log_{10} 7$, $a_\eta = 4813$, the above condition $a_\eta > 12M(N)$ holds, because $M(N) = 13$ if $N \in [q_{\eta+1}, q_{\eta+2})$, $M(N) = 15$ if $N \in [q_{\eta+2}, q_{\eta+2} + q_\eta)$ or if $N \in [q_{\eta+2} + q_{\eta+1}, 2q_{\eta+2})$. By using (17), we can explain "quadratic-function" like behaviors observed not only for $\alpha = 1 - \log_{10} 7$, but also for $\alpha = 2 - \log_{10} 33$ and $\alpha = 2 - \log_{10} 54$. Because for these α's, the conditions (15) and (16) are satisfied and it implies that the "peak" of each "hill" is much larger than the fluctuation $2M(N)$ at the valley's,

$$a_\eta f\left(\frac{1}{2}\right) - M(N) \fallingdotseq \frac{a_\eta}{4} - M(N) > 2M(N).$$

Since for general α, satisfying the condition (15), $a_\eta f(1/2)$ is sufficiently large, we can see that each "peak" is strictly positive and sufficiently large.

Next, we consider the estimates for $ND_N^*(n\alpha)$, restricted to a specific irrational number α, $\alpha = 1 - \log_{10} 7$, and specific values of N, having coefficients $b_0 = b_1 = b_2 = b_3 = b_4 = 0$ and $0 \le b_{\eta-1} \le 4813$.

Theorem 4.5. *Let $\alpha = 1 - \log_{10} 7$. Assume that N has coefficients $b_0 = b_1 = b_2 = b_3 = b_4 = 0$ and $0 \le b_{\eta-1} \le 4813$. Then we have*

$$-4 < ND_N^*(n\alpha) - b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) < 9 \tag{18}$$

for $N \in [q_{\eta+1}, q_{\eta+2})$ and $[q_{\eta+2}+q_{\eta+1}, 2q_{\eta+2})$. Note that 9 and -4 are the common bounds over these two intervals. Moreover, we have (18) with 9 replaced by 8 for $N \in [q_{\eta+2}, q_{\eta+2} + q_\eta)$.

Remark 3. The estimate in Theorem 4.5 can be rewritten as follows:

$$-4 < ND_N^*(n\alpha) - a_\eta f(x') < 9 \quad \text{for } x' = \frac{N'}{a_\eta q_{\eta-1}},$$

where $N' = N - q_{\eta+1}$ if $N \in [q_{\eta+1}, q_{\eta+2})$, $N' = N - q_{\eta+2}$ if $N \in [q_{\eta+2}, q_{\eta+2} + q_\eta)$ and $N' = N - (q_{\eta+2} + q_{\eta+1})$ if $N \in [q_{\eta+2} + q_{\eta+1}, 2q_{\eta+2})$. This estimate is not whole estimate, but this gives a still more precise explanation of "quadratic-function" like behavior of $ND_N^*(n\alpha)$, $\alpha = 1 - \log_{10} 7$.

5. Some lemmas and proof of Corollary 3.2

In this section we first prepare some lemmas for the proofs of Corollary 3.2, Theorems 4.3, 4.4 and 4.5, and then we prove Corollary 3.2.

We first estimate the terms $b_{2j}(1 - q_{2j}A_{2j})$ and $b_{2j+1}(1 + q_{2j+1}A_{2j+1})$ in Theorem 3.1, (7).

Lemma 5.1. *Let α be a positive irrational number. Then for $0 \le j \le [m/2]$ we have*

$$0 \le b_{2j}(1 - q_{2j}A_{2j}) \le b_{2j}, \tag{19}$$

with equality if and only if $b_{2j} = 0$. Moreover, for $0 \le j \le [(m-1)/2]$ we have

$$0 \le b_{2j+1}(1 + q_{2j+1}A_{2j+1}) \le b_{2j+1}, \tag{20}$$

with equality if and only if $b_{2j+1} = 0$.

Proof. In case $b_{2j} = 0$, (19) is trivial. In case $b_{2j} \ne 0$, we have $0 < q_{2j}A_{2j} < 1$ by Lemma 2.1 (i) and (iii). Hence we obtain (19). We can also prove (20) in the same way as (19). □

The next lemma gives the estimates for terms $b_{2j}(1 - q_{2j}A_{2j})$ and $b_{2j+1}(1 + q_{2j+1}A_{2j+1})$ in Theorem 3.1, (7), by using terms $b_{2j}(1-b_{2j}/a_{2j+1})$ and $b_{2j+1}(1-b_{2j+1}/a_{2j+2})$, respectively.

Lemma 5.2. *Let α be a positive irrational number. Then for $0 \le j \le [m/2]$ we have*

$$b_{2j}\left(1 - \frac{b_{2j}}{a_{2j+1}}\right) - 2 < b_{2j}(1 - q_{2j}A_{2j}) < b_{2j}\left(1 - \frac{b_{2j}}{a_{2j+1}}\right) + 3. \tag{21}$$

Moreover, for $0 \le j \le [(m-1)/2]$ we have

$$b_{2j+1}\left(1 - \frac{b_{2j+1}}{a_{2j+2}}\right) - 2 < b_{2j+1}(1 + q_{2j+1}A_{2j+1}) < b_{2j+1}\left(1 - \frac{b_{2j+1}}{a_{2j+2}}\right) + 3. \tag{22}$$

Proof. We first prove (21). For the simplicity of the notation, we consider $b_j(1-q_jA_j)$ for even j, $0 \le j \le m$, instead of $b_{2j}(1 - q_{2j}A_{2j})$ for $0 \le j \le [m/2]$. The case $b_j = 0$ is trivial. Let us consider the case where $0 < b_j \le a_{j+1}$. By definition of A_j, we can write $b_j(1 - q_jA_j)$ in the form

$$\begin{aligned} b_j(1 - q_jA_j) = b_j(1 - b_jq_j(q_j\alpha - p_j)) \\ - b_jq_jN_{j-1}(\alpha - r_j) - b_jq_j \sum_{t=j+1}^{m} b_t(q_t\alpha - p_t). \end{aligned} \tag{23}$$

First, we estimate the first term in the right-hand side of (23) by using term $b_j(1 - b_j/a_{j+1})$. Let us denote $\zeta_n = [a_n; a_{n+1}, a_{n+2}, \ldots]$. Using basic facts that $q_n\alpha - p_n = (-1)^n/(q_n\zeta_{n+1} + q_{n-1})$ and $q_{n-1}/q_n = [0; a_n, a_{n-1}, \ldots, a_1]$ (cf. [8, pp.9-10]), we have

$$b_jq_j(q_j\alpha - p_j) = \frac{b_j}{[a_{j+1}; a_{j+2}, a_{j+3}, \ldots] + [0; a_j, a_{j-1}, \ldots, a_1]}.$$

Since $a_{j+1} < [a_{j+1}; a_{j+2}, a_{j+3}, \ldots] + [0; a_j, a_{j-1}, \ldots, a_1] < a_{j+1} + 2$, we see that

$$\frac{b_j}{a_{j+1}} > b_j q_j (q_j \alpha - p_j) > \frac{b_j}{a_{j+1} + 2}.$$

This right-hand side equals $b_j/a_{j+1} - 2b_j/a_{j+1}(a_{j+1}+2)$, and $2b_j^2/a_{j+1}(a_{j+1}+2)$ is less than 2. Hence we obtain

$$b_j \left(1 - \frac{b_j}{a_{j+1}}\right) < b_j \left(1 - b_j q_j (q_j \alpha - p_j)\right) < b_j \left(1 - \frac{b_j}{a_{j+1}}\right) + 2. \tag{24}$$

Next, we estimate the second term in the right-hand side of (23). Since $0 \le N_{j-1} < q_j$, and since $\alpha - r_j > 0$ for even j, we have

$$0 \le b_j q_j N_{j-1}(\alpha - r_j) < b_j q_j |q_j \alpha - p_j|$$

(recall that $q_j(\alpha - r_j) = q_j \alpha - p_j$). From (2) and $b_j q_j \le a_{j+1} q_j \le q_{j+1}$, we find $b_j q_j |q_j \alpha - p_j| < 1$ and therefore

$$-1 < -b_j q_j N_{j-1}(\alpha - r_j) \le 0. \tag{25}$$

Finally, we estimate the last term in the right-hand side of (23). Using Lemma 2.1 (ii) for $t = j+1, j+2, \ldots, m$ instead of $t = j, j+1, \ldots, m$, we have

$$-b_j q_j |q_{j+1}\alpha - p_{j+1}| < (-1)^{j+1} b_j q_j \sum_{t=j+1}^{m} b_t (q_t \alpha - p_t) < b_j q_j |q_j \alpha - p_j|.$$

Since $b_j q_j |q_{j+1}\alpha - p_{j+1}| < 1$, and using $b_j q_j |q_j \alpha - p_j| < 1$ once more, we have

$$-1 < -b_j q_j \sum_{t=j+1}^{m} b_t (q_t \alpha - p_t) < 1. \tag{26}$$

Combining (24) - (26) with (23), we obtain (21). We can also prove (22) in the same way as in the above. □

Now, we return back to the proof of Corollary 3.2. To derive Corollary 3.2 from Theorem 3.1, we use this Lemma 5.2.

Proof of Corollary 3.2. We first estimate $ND_N^*(n\alpha)$ from above. We estimate the sums $\sum b_{2j}(1 - q_{2j}A_{2j})$ and $\sum b_{2j+1}(1 + q_{2j+1}A_{2j+1})$ in Theorem 3.1, (7). Since $b_{2j}(1 - q_{2j}A_{2j}) < b_{2j}(1 - b_{2j}/a_{2j+1}) + 3$ (cf. Lemma 5.2, (21)), we have

$$\sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) < \sum_{j=0}^{[m/2]} b_{2j}\left(1 - \frac{b_{2j}}{a_{2j+1}}\right) + 3c_1,$$

where $c_1 = [m/2]+1$. On the other hand, since $b_{2j+1}(1+q_{2j+1}A_{2j+1}) < b_{2j+1}(1-b_{2j+1}/a_{2j+2}) + 3$ (cf. Lemma 5.2, (22)), we also have

$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) < \sum_{j=0}^{[(m-1)/2]} b_{2j+1}\left(1-\frac{b_{2j+1}}{a_{2j+2}}\right) + 3c_2,$$

where $c_2 = [(m-1)/2]+1$. By using these two estimates and Theorem 3.1, (7), we obtain

$$\begin{aligned} &ND_N^*(n\alpha) \\ &< \max\left(\sum_{j=0}^{[m/2]} b_{2j}\left(1-\frac{b_{2j}}{a_{2j+1}}\right), \sum_{j=0}^{[(m-1)/2]} b_{2j+1}\left(1-\frac{b_{2j+1}}{a_{2j+2}}\right) + A_0\right) + c + 3c_1. \end{aligned}$$

Since $A_0 < 1$ (cf. Lemma 2.1 (iii)) and the error term c in Theorem 3.1, (7), satisfies $c < [(m+1)/2]+1$, we obtain the upper estimate for $ND_N^*(n\alpha)$.

The lower estimate for $ND_N^*(n\alpha)$ can be shown by similar arguments. Since $b_{2j}(1-q_{2j}A_{2j}) > b_{2j}(1-b_{2j}/a_{2j+1}) - 2$ and $b_{2j+1}(1+q_{2j+1}A_{2j+1}) > b_{2j+1}(1-b_{2j+1}/a_{2j+2}) - 2$ (cf. Lemma 5.2, (21) and (22)) we have

$$\sum_{j=0}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}) > \sum_{j=0}^{[m/2]} b_{2j}\left(1-\frac{b_{2j}}{a_{2j+1}}\right) - 2c_1,$$

and

$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) > \sum_{j=0}^{[(m-1)/2]} b_{2j+1}\left(1-\frac{b_{2j+1}}{a_{2j+2}}\right) - 2c_2,$$

where $c_1 = [m/2]+1$ and $c_2 = [(m-1)/2]+1$. From these two estimates and Theorem 3.1, (7), we obtain

$$\begin{aligned} &ND_N^*(n\alpha) \\ &> \max\left(\sum_{j=0}^{[m/2]} b_{2j}\left(1-\frac{b_{2j}}{a_{2j+1}}\right), \sum_{j=0}^{[(m-1)/2]} b_{2j+1}\left(1-\frac{b_{2j+1}}{a_{2j+2}}\right) + A_0\right) + c - 2c_1. \end{aligned}$$

Since $A_0 > -1$ (cf. Lemma 2.1 (iii)) and $c \geq -1$, we obtain the lower estimate for $ND_N^*(n\alpha)$. $\square$

6. Proofs for results of Section 4

In this section we give the proofs for results of Section 4. We first begin with the proof of Theorem 4.1.

Proof of Theorem 4.1. Throughout the proof, we prove only the case where m is even. The proof for the case where m is odd, is done similarly. The proof is based on Theorem B. As for the first sum and the maximum term in the right-hand side of Theorem B, (6), we note that

$$\sum_{j=0}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}) - \sum_{j=0}^{m} b_j((-1)^j - q_jA_j) = \sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}).$$

(i) By the assumption $N = b_mq_m + b_{m-2}q_{m-2}$, $1 \le b_m \le a_{m+1}$, $1 \le b_{m-2} \le a_{m-1}$, we have $b_0 = b_1 = \cdots = b_{m-3} = 0$ and $b_{m-1} = 0$. From this, we have the following:

$$\begin{aligned} &\sum_{j=0}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}) = b_{m-2}(1-q_{m-2}A_{m-2}) + b_m(1-q_mA_m), \\ &\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) = 0. \end{aligned} \tag{27}$$

We now consider the summations, such as, the form $\sum a_{j+1}q_jA_j$ in Theorem B, (6). As for A_j, observe that $A_0 = A_1 = \cdots = A_{m-2} = b_{m-2}q_{m-2}(\alpha - r_{m-2}) + b_mq_m(\alpha - r_m) > 0$ and that $A_m = b_{m-2}q_{m-2}(\alpha - r_m) + b_mq_m(\alpha - r_m) > 0$, because $\alpha - r_{m-2} > 0$ and $\alpha - r_m > 0$. Notice that, however, the sign of $A_{m-1} = b_{m-2}q_{m-2}(\alpha - r_{m-1}) + b_mq_m(\alpha - r_m)$ varies depending on α, b_m and b_{m-2}, because $\alpha - r_{m-1} < 0$. Thus we need to consider the cases $A_{m-1} > 0$ and $A_{m-1} < 0$ separately.

Case 1. If $A_{m-1} > 0$, then the above results for A_j imply that $A_j > 0$ for any j, $0 \le j \le m$, and it suggests that $\{j : j \in P \text{ is odd},\ 1 \le j \le m\} = \{1, 3, \ldots, m-1\}$. Therefore we have

$$\sum_{\substack{j \text{ odd} \\ j \in P}} a_{j+1}q_jA_j = q_mA_m - A_0, \tag{28}$$

where we used (4). On the other hand, we have $\{j : j \notin P \text{ is even},\ 0 \le j \le m\} = \emptyset$ and hence

$$\sum_{\substack{j \text{ even} \\ j \notin P}} a_{j+1}q_jA_j = 0. \tag{29}$$

Combining (27) - (29) with Theorem B, (6), we obtain (put $K_j = (-1)^j q_j A_j$)

$$\begin{aligned}&ND_N^*(n\alpha)\\&= \max(b_{m-2}(1-q_{m-2}A_{m-2}) + b_m(1-q_mA_m), A_0) + q_mA_m - A_0 + \varepsilon\\&= \max(b_{m-2}(1-q_{m-2}A_{m-2}) + b_m(1-q_mA_m) - |A_0|, 0) + q_mA_m + \varepsilon\\&= \max(b_{m-2}(1-K_{m-2}) + b_m(1-K_m) - |A_0|, 0) + K_m + \varepsilon.\end{aligned}$$

Case 2. If $A_{m-1} < 0$, then we have $\{j : j \in P \text{ is odd}, \ 1 \le j \le m\} = \{1, 3, \ldots, m-3\}$ and hence

$$\sum_{\substack{j \text{ odd}\\ j\in P}} a_{j+1}q_jA_j = q_{m-2}A_{m-2} - A_0,$$

where we used (4). On the other hand, we again have (29), since $\{j : j \notin P \text{ is even}, \ 0 \le j \le m\} = \emptyset$. By combining these with Theorem B, (6), we obtain

$$\begin{aligned}&ND_N^*(n\alpha)\\&= \max(b_{m-2}(1-q_{m-2}A_{m-2}) + b_m(1-q_mA_m), A_0) + q_{m-2}A_{m-2} - A_0 + \varepsilon\\&= \max(b_{m-2}(1-q_{m-2}A_{m-2}) + b_m(1-q_mA_m) - |A_0|, 0) + q_{m-2}A_{m-2} + \varepsilon\\&= \max(b_{m-2}(1-K_{m-2}) + b_m(1-K_m) - |A_0|, 0) + K_{m-2} + \varepsilon.\end{aligned}$$

(ii) By the assumption $N = b_mq_m + b_{m-1}q_{m-1}$, $1 \le b_m < a_{m+1}$, $1 \le b_{m-1} \le a_m$, we have $b_0 = b_1 = \cdots = b_{m-2} = 0$. From this, we have the following:

$$\begin{aligned}&\sum_{j=0}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}) = b_m(1-q_mA_m),\\&\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) = b_{m-1}(1+q_{m-1}A_{m-1}).\end{aligned}\tag{30}$$

In this case, we have $A_0 = A_1 = \cdots = A_{m-1} = b_{m-1}q_{m-1}(\alpha - r_{m-1}) + b_mq_m(\alpha - r_m) < 0$ because of Lemma 2.1 (i) and $b_{m-1} \ne 0$. Similarly, $A_m = b_{m-1}q_{m-1}(\alpha - r_m) + b_mq_m(\alpha - r_m) > 0$, since $b_m \ne 0$. From these relations, we see that $\{j : j \in P \text{ is odd}, \ 1 \le j \le m\} = \emptyset$, hence that

$$\sum_{\substack{j \text{ odd}\\ j\in P}} a_{j+1}q_jA_j = 0.\tag{31}$$

On the other hand, we have $\{j : j \notin P \text{ is even}, \ 0 \le j \le m\} = \{0, 2, \ldots, m-2\}$ and therefore

$$\sum_{\substack{j \text{ even} \\ j \notin P}} a_{j+1} q_j A_j = q_{m-1} A_{m-1}, \tag{32}$$

where we used (4). Combining (30) - (32) with Theorem B, (6), we obtain

$$\begin{aligned} &N D_N^*(n\alpha) \\ &= \max(b_m(1 - q_m A_m), b_{m-1}(1 + q_{m-1} A_{m-1}) + A_0) - q_{m-1} A_{m-1} + \varepsilon \\ &= \max(b_m(1 - q_m A_m), b_{m-1}(1 + q_{m-1} A_{m-1}) - |A_0|) - q_{m-1} A_{m-1} + \varepsilon \\ &= \max(b_m(1 - K_m), b_{m-1}(1 - K_{m-1}) - |A_0|) + K_{m-1} + \varepsilon. \end{aligned}$$

□

Remark 4. By Lemma 2.1 (i) and (iii), under the conditions of Theorem 4.1 (i), we obtain $0 < (-1)^m A_m < 1/q_m$ and $0 < (-1)^{m-2} A_{m-2} < 1/q_{m-2}$. Therefore we have $0 < K_m < 1$ and $0 < K_{m-2} < 1$. Similarly, under the conditions of Theorem 4.1 (ii), we have $0 < K_m < 1$ and $0 < K_{m-1} < 1$.

Next, we prove Theorem 4.2 by using Corollary C and Theorem 4.1.

Proof of Theorem 4.2. (i) Assume that $N = b_m q_m$, $1 \le b_m \le a_{m+1}$. Then it is easy to see from $b_m \ge 1$ that $(b_m - 1) b_m q_m |q_m \alpha - p_m| \ge 0$ and $b_m |q_m \alpha - p_m| > 0$. Therefore Corollary C, (10), implies that

$$N D_N^*(n\alpha) < b_m.$$

(ii) Assume that $N = b_m q_m + b_{m-2} q_{m-2}$, $1 \le b_m \le a_{m+1}$, $1 \le b_{m-2} \le a_{m-1}$. Then we have $0 < K_m < 1$ and $0 < K_{m-2} < 1$ (cf. Remark 4). Here $K_j = (-1)^j q_j A_j$. We discuss the cases $(-1)^{m-1} A_{m-1} > 0$ and $(-1)^{m-1} A_{m-1} < 0$ separately. If $(-1)^{m-1} A_{m-1} > 0$, then we have by Theorem 4.1, (11),

$$\begin{aligned} N D_N^*(n\alpha) &< \max(b_m + b_{m-2} - K_{m-2} - |A_0|, 0) + K_{m-2} + \varepsilon \\ &= \max(b_m + b_{m-2} - |A_0|, K_{m-2}) + \varepsilon \\ &< b_m + b_{m-2} + 1, \end{aligned}$$

where we used $b_m \ge 1$, $b_{m-2} \ge 1$, $|A_0| < 1$ and $|\varepsilon| \le 1$. In case $(-1)^{m-1} A_{m-1} < 0$, the upper bound can also be shown in the same way as in the above.

(iii) Assume that $N = b_m q_m + b_{m-1} q_{m-1}$, $1 \le b_m < a_{m+1}$, $1 \le b_{m-1} \le a_m$. Then we have $0 < K_m < 1$ and $0 < K_{m-1} < 1$ (cf. Remark 4). Therefore, by

Theorem 4.1, (12), we obtain

$$\begin{aligned} ND_N^*(n\alpha) &< \max\left(b_m - K_m, b_{m-1} - K_{m-1} - |A_0|\right) + K_{m-1} + \varepsilon \\ &= \max\left(b_m - K_m + K_{m-1}, b_{m-1} - |A_0|\right) + \varepsilon \\ &< \max(b_m + 1, b_{m-1}) + 1, \end{aligned}$$

where we used $b_m \geq 1$, $b_{m-1} \geq 1$, $|A_0| < 1$ and $|\varepsilon| \leq 1$. □

In the latter part of this section, we give the proofs of Theorems 4.3 and 4.4 only for the case $\alpha = 1 - \log_{10} 7$. The proofs are valid for other irrational numbers with an isolated large partial quotient, with slight modifications.

Before starting proofs, we first explain briefly how to prove these theorems. The proofs are based on Theorem 3.1. For a special term $b_{\eta-1}(1 + q_{\eta-1}A_{\eta-1})$ in Theorem 3.1, (7), $0 \leq b_{\eta-1} \leq 4813$, we use the following estimate:

$$b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) - 2 < b_{\eta-1}(1 + q_{\eta-1}A_{\eta-1}) < b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 3 \qquad (33)$$

by Lemma 5.2, (22). For the other terms $b_{2j}(1-q_{2j}A_{2j})$ and $b_{2j+1}(1+q_{2j+1}A_{2j+1})$ in Theorem 3.1, (7), we apply Lemma 5.1, (19) and (20), respectively.

Proof of Theorem 4.3. Let $\alpha = 1 - \log_{10} 7$. Assume that $N \in [q_\eta, q_{\eta+1})$. Then we have $m = \eta$ and $b_\eta = 1$. Since $b_\eta = a_{\eta+1}$, we have $b_{\eta-1} = 0$ (recall that $b_{j-1} = 0$ if $b_j = a_{j+1}$). Therefore $b_{\eta-1}(1 + q_{\eta-1}A_{\eta-1}) = 0$. The other coefficients b_j take values $0 \leq b_j \leq a_{j+1}$. Using Lemma 5.1, (19) and (20), we can estimate the sums $\sum b_{2j}(1 - q_{2j}A_{2j})$ and $\sum b_{2j+1}(1 + q_{2j+1}A_{2j+1})$ in Theorem 3.1, (7), as follows:

$$\sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) < \sum_{\substack{j=0 \\ j \neq (\eta-1)/2}}^{[m/2]} b_{2j} \leq \sum_{\substack{j=0 \\ j \neq (\eta-1)/2}}^{[m/2]} a_{2j+1},$$

$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1 + q_{2j+1}A_{2j+1}) < \sum_{\substack{j=0 \\ j \neq (\eta-2)/2}}^{[(m-1)/2]} b_{2j+1} \leq \sum_{\substack{j=1 \\ j \neq \eta/2}}^{[(m+1)/2]} a_{2j}.$$

By using these estimates and Theorem 3.1, (7), we obtain

$$ND_N^*(n\alpha) < M(N) + |A_0| + c < M(N) + \left[\frac{m+1}{2}\right] + 2,$$

where we used $|A_0| < 1$ (cf. Lemma 2.1 (iii)) and $c < [(m+1)/2] + 1$. By the condition (13), $M(N) \geq [(m+1)/2] + 2$, we obtain upper bound $2M(N)$ in Theorem 4.3.

In case $N \in [q_{\eta+2} + q_\eta, q_{\eta+2} + q_{\eta+1})$, we can show the upper bound $2M(N)$ by similar arguments. $\square$

Proof of Theorem 4.4. Let $\alpha = 1 - \log_{10} 7$. Assume that $N \in [q_{\eta+1}, q_{\eta+2})$. Then we have $m = \eta + 1$ and $b_{\eta+1} = 1$. Since $b_{\eta+1} = a_{\eta+2}$, we have $b_\eta = 0$, that is $b_\eta \neq a_{\eta+1}$. The essential difference between this result and Theorem 4.3 is that the special coefficient $b_{\eta-1}$ takes values $0 \leq b_{\eta-1} \leq 4813$ (recall that the value of the coefficient $b_{\eta-1}$ is always 0 under the conditions of Theorem 4.3, because $b_\eta = a_{\eta+1}$).

We begin by showing upper estimate for $ND_N^*(n\alpha)$. By Lemma 5.1, (19), we can estimate the sum $\sum b_{2j}(1 - q_{2j}A_{2j})$ in Theorem 3.1, (7), as follows:

$$\sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) < \sum_{\substack{j=0 \\ j \neq (\eta-1)/2}}^{[m/2]} a_{2j+1}.$$

On the other hand, as for $b_{\eta-1}(1 + q_{\eta-1}A_{\eta-1})$, we use the upper bound of (33), and we apply Lemma 5.1, (20), to the other terms $b_{2j+1}(1 + q_{2j+1}A_{2j+1})$. As a result, we have

$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1 + q_{2j+1}A_{2j+1}) < \sum_{\substack{j=1 \\ j \neq \eta/2}}^{[(m+1)/2]} a_{2j} + b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 3.$$

By using these two estimates and Theorem 3.1, (7), we obtain

$$\begin{aligned} ND_N^*(n\alpha) &< b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 3 + M(N) + |A_0| + c \\ &< b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + M(N) + \left[\frac{m+1}{2}\right] + 5, \end{aligned}$$

where we used $|A_0| < 1$ (cf. Lemma 2.1 (iii)) and $c < [(m+1)/2] + 1$ in the last inequality. It is easy to see that $M(N) \geq [(m+1)/2]$ holds for general irrational numbers, because $a_j > 0$ for $1 \leq j \leq m$. Moreover, in this case $N \in [q_{\eta+1}, q_{\eta+2})$, the condition (16), $M(N) \geq 5$, hold, because $M(N) = 13$. Therefore, we obtain upper bound $3M(N)$ in Theorem 4.4.

The lower estimate for $ND_N^*(n\alpha)$ can be shown by similar arguments. By using the lower bound of (33), and Lemma 5.1, (19) and (20), we have the following:

$$\sum_{j=0}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}) \geq 0,$$

$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) > b_{\eta-1}\left(1-\frac{b_{\eta-1}}{a_\eta}\right) - 2.$$

From these two estimates and Theorem 3.1, (7), we obtain

$$ND_N^*(n\alpha) > b_{\eta-1}\left(1-\frac{b_{\eta-1}}{a_\eta}\right) - 2 - |A_0| + c > b_{\eta-1}\left(1-\frac{b_{\eta-1}}{a_\eta}\right) - 4,$$

where we used $|A_0| < 1$ (cf. Lemma 2.1 (iii)) and $c \geq -1$ in the last inequality. The condition (16), $M(N) \geq 5$, of course, mean that $-M(N) < -4$, and consequently, we obtain lower bound $-M(N)$ in Theorem 4.4.

In both cases $N \in [q_{\eta+2}, q_{\eta+2}+q_\eta)$ and $N \in [q_{\eta+2}+q_{\eta+1}, 2q_{\eta+2})$, the coefficient $b_{\eta-1}$ takes values $0 \leq b_{\eta-1} \leq 4813$, because $b_\eta \neq a_{\eta+1}$. Moreover, since $M(N) = 15$, the condition (16) is satisfied. Thus, for these two intervals, this theorem can be shown in the same way as in the above. □

Proof of Theorem 4.5. The proof for the lower bound goes similarly as the proof in Theorem 4.4, and we prove only the upper bound.

Assume first that $N \in [q_{\eta+1}, q_{\eta+2})$ and that N has coefficients $b_0 = b_1 = b_2 = b_3 = b_4 = 0$ and $0 \leq b_{\eta-1} \leq 4813$. Then $m = \eta+1$, $b_{\eta+1} = 1$ and $b_\eta = 0$ because $b_{\eta+1} = a_{\eta+2}$. Hence the sums $\sum b_{2j}(1-q_{2j}A_{2j})$ and $\sum b_{2j+1}(1+q_{2j+1}A_{2j+1})$ in Theorem 3.1, (7), are the following:

$$\sum_{j=0}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}) = 0,$$

$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) = b_{\eta-1}(1+q_{\eta-1}A_{\eta-1}) + b_{\eta+1}(1+q_{\eta+1}A_{\eta+1}).$$

As for $b_{\eta-1}(1+q_{\eta-1}A_{\eta-1})$, we use the upper bound of (33). On the other hand, we have $b_{\eta+1}(1+q_{\eta+1}A_{\eta+1}) < 1$ from Lemma 5.1, (20). Also, since $\alpha - r_{\eta-1} < 0$ and $\alpha - r_{\eta+1} < 0$, we have $A_0 = b_{\eta-1}q_{\eta-1}(\alpha - r_{\eta-1}) + b_{\eta+1}q_{\eta+1}(\alpha - r_{\eta+1}) < 0$. Thus by Theorem 3.1, (7), we obtain

$$ND_N^*(n\alpha) < \max\left(0, b_{\eta-1}\left(1-\frac{b_{\eta-1}}{a_\eta}\right) + 4 + A_0\right) + c < b_{\eta-1}\left(1-\frac{b_{\eta-1}}{a_\eta}\right) + 9.$$

The last inequality results from $A_0 < 0$ and $c < [(m+1)/2] + 1$.

Next, assume that $N \in [q_{\eta+2}, q_{\eta+2} + q_\eta)$ and that N has coefficients $b_0 = b_1 = b_2 = b_3 = b_4 = 0$ and $0 \le b_{\eta-1} \le 4813$. Then $m = \eta + 2$, $b_{\eta+2} = 1$ and $b_{\eta+1} = b_\eta = 0$. Therefore we have the following:

$$\sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) = b_{\eta+2}(1 - q_{\eta+2}A_{\eta+2}),$$
$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1 + q_{2j+1}A_{2j+1}) = b_{\eta-1}(1 + q_{\eta-1}A_{\eta-1}).$$

In case $b_{\eta-1} = 0$, Theorem 4.5 is trivial. We, therefore, consider only the case where $0 < b_{\eta-1} \le 4813$. By Lemma 5.1, (19), we have $b_{\eta+2}(1 - q_{\eta+2}A_{\eta+2}) < 1$. Observe that $A_0 = A_{\eta-1} = b_{\eta-1}q_{\eta-1}(\alpha - r_{\eta-1}) + b_{\eta+2}q_{\eta+2}(\alpha - r_{\eta+2})$. Hence $b_{\eta-1} \ne 0$ implies that $A_0 < 0$ from Lemma 2.1 (i). By the same way as $N \in [q_{\eta+1}, q_{\eta+2})$, we obtain

$$ND_N^*(n\alpha) < \max\left(1, b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 3 + A_0\right) + c < b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 8.$$

The last inequality results from $A_0 < 0$ and $c < [(m+1)/2] + 1$.

Finally, assume that $N \in [q_{\eta+2} + q_{\eta+1}, 2q_{\eta+2})$ and that N has coefficients $b_0 = b_1 = b_2 = b_3 = b_4 = 0$ and $0 \le b_{\eta-1} \le 4813$. Then $m = \eta + 2$, $b_{\eta+2} = b_{\eta+1} = 1$ and $b_\eta = 0$ because $b_{\eta+1} = a_{\eta+2}$. Thus we have the following:

$$\sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) = b_{\eta+2}(1 - q_{\eta+2}A_{\eta+2}),$$
$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1 + q_{2j+1}A_{2j+1}) = b_{\eta-1}(1 + q_{\eta-1}A_{\eta-1}) + b_{\eta+1}(1 + q_{\eta+1}A_{\eta+1}).$$

By similar arguments to the proof of the cases $N \in [q_{\eta+1}, q_{\eta+2})$ and $N \in [q_{\eta+2}, q_{\eta+2}+q_\eta)$, we can show that $b_{\eta+2}(1-q_{\eta+2}A_{\eta+2}) < 1$, $b_{\eta+1}(1+q_{\eta+1}A_{\eta+1}) < 1$ and $A_0 < 0$. Therefore we obtain

$$ND_N^*(n\alpha) < \max\left(1, b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 4 + A_0\right) + c < b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 9.$$

The last inequality results from $A_0 < 0$ and $c < [(m+1)/2] + 1$. □

Remark 5. These proofs are valid for $2 - \log_{10} 33$ and $2 - \log_{10} 54$, with some obvious modifications on $\eta = 3$ and $\eta = 7$, respectively, in the above.

7. Other examples

In this section we first report the extraordinary behaviors, observed in cases of $2 - \log_{10} 33$ and $2 - \log_{10} 54$. The continued fraction expansion of $2 - \log_{10} 33$ is given by

$$2 - \log_{10} 33 = [0; 2, 13, 299, 1, 1, 10, \ldots],$$

and $2 - \log_{10} 33$ has a rather large isolated 3rd partial quotient, $a_3 = 299$, and the 3rd-order convergent, $q_3 = 8075$. Since $M(N) = 14$ if $m = \eta$ or if $m = \eta + 1$, and $M(N) = 24$ if $m = \eta + 2$, $2 - \log_{10} 33$ satisfies the conditions (13), (15) and (16).

The continued fraction expansion of $2 - \log_{10} 54$ is given by

$$2 - \log_{10} 54 = [0; 3, 1, 2, 1, 3, 1, 326, 2, 1, 3, \ldots],$$

and $2 - \log_{10} 54$ also has a rather large isolated 7th partial quotient, $a_7 = 326$, and the 7th-order convergent, $q_7 = 23202$. Since $M(N) = 8$ if $m = \eta$, and $M(N) = 9$ if $m = \eta + 1$ or if $m = \eta + 2$, $2 - \log_{10} 54$ also satisfies the conditions (13), (15) and (16).

Although these large partial quotients, 299 and 326, are not so large in comparison with the quotient 4813 in case of $1 - \log_{10} 7$, we can, again in each case, observe extraordinary phenomena similar to the phenomenon, shown in case of $1 - \log_{10} 7$ (see Figs. 2 and 3).

Next, we consider the numerical estimation of $ND_N^*(n\alpha)$, in case of $\log_{10} 2$, $\log_{10} 3$ and $1 - \log_{10} 5$, continued fraction expansions are given by

$$\begin{aligned}
\log_{10} 2 &= [0; 3, 3, 9, 2, 2, 4, 6, 2, 1, 1, 3, 1, 18, 1, 6, 1, 2, 1, 1, 4, \ldots],\\
\log_{10} 3 &= [0; 2, 10, 2, 2, 1, 13, 1, 7, 18, 2, 2, 1, 2, 3, 4, 1, 1, 14, 2, 44, \ldots],\\
1 - \log_{10} 5 &= [0; 3, 3, 9, 2, 2, 4, 6, 2, 1, 1, 3, 1, 18, 1, 6, 1, 2, 1, 1, 4, \ldots].
\end{aligned}$$

We do not find large partial quotients in continued fraction expansions within 20 partial quotients. In case of $\log_{10} 2$, the value of $D_N^*(n\alpha)$ converge to 0 very quickly (cf. Fig. 4); when $N = 3{,}000{,}000$, the value of $D_N^*(n\alpha)$ equals a extraordinarily small value, 2.7×10^{-6}. In case of, similarly, $\log_{10} 3$ and $1 - \log_{10} 5$, values of $D_N^*(n\alpha)$ converge very quickly to 0. In contrast with Fig. 1, Fig. 4 shows “very normal” decay of $D_N^*(n\alpha)$.

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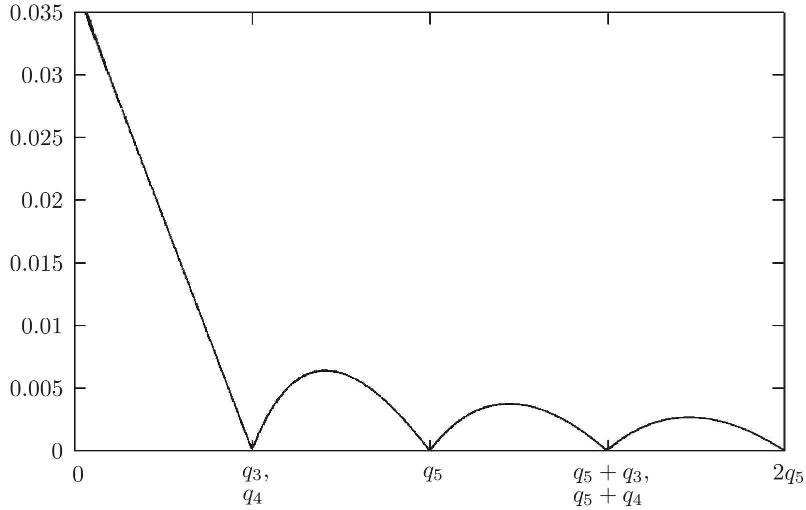

FIGURE 2. $D_N^*(n\alpha)$, $\alpha = 2 - \log_{10} 33$, up to $N = 2q_5$, every 10 points. $q_3 = 8{,}075$, $q_4 = 8{,}102$, $q_5 = 16{,}177$, $q_5 + q_3 = 24{,}252$, $q_5 + q_4 = 24{,}279$, $2q_5 = 32{,}354$.

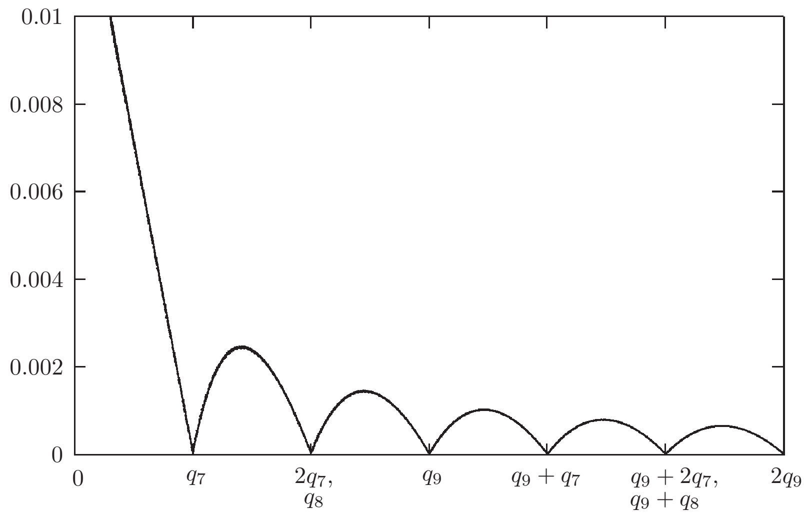

FIGURE 3. $D_N^*(n\alpha)$, $\alpha = 2 - \log_{10} 54$, up to $N = 2q_9$, every 10 points. $q_7 = 23{,}202$, $2q_7 = 46{,}404$, $q_8 = 46{,}475$, $q_9 = 69{,}677$, $q_9 + q_7 = 92{,}879$, $q_9 + 2q_7 = 116{,}081$, $q_9 + q_8 = 116{,}152$, $2q_9 = 139{,}354$.

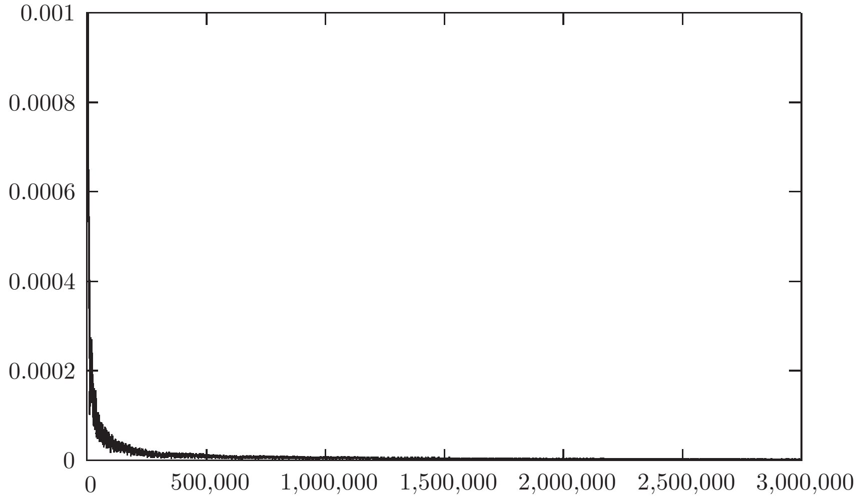

FIGURE 4. $D_N^*(n\alpha)$, $\alpha = \log_{10} 2$, up to $N = 3{,}000{,}000$, every 100 points.

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