ON THE DISCREPANCY OF IRRATIONAL ROTATIONS WITH ISOLATED LARGE PARTIAL QUOTIENTS: LONG TERM EFFECTS*

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Abstract. Setokuchi and Takashima [7] gave general mathematical explanations for the emergence of several parabola-like hills in the behavior of the discrepancies of irrational rotations having single isolated large partial quotient, in somewhat short range of N. We extend estimates in [7] and give some general conditions which ensure repetitions of hills in much longer range of N. For example, in case of $1-\log_{10} 7$, our conditions show that more than 2.7×10^{27} repetitions of hills exist, where N runs up to over 6.8×10^{33}.

1. Introduction

For an irrational number $\alpha \in (0,1)$, the sequence $\{n\alpha\}$, $n \in \mathbb{N}$, is well known to be uniformly distributed mod 1, and its discrepancy $D_N(n\alpha)$ defined by

$$D_N(n\alpha) = \sup_{0 \le a < b \le 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[a,b)}(\{n\alpha\}) - (b-a) \right|,$$

tends to zero as $N \to \infty$, where $\{n\alpha\}$ denotes the fractional part of $n\alpha$, and $\mathbf{1}_{[a,b)}$ denotes the indicator function of the interval $[a,b)$ (cf. e.g. Drmota and Tichy [1], Kuipers and Niederreiter [3] and Weyl [8]).

Kesten [2] shows the following limit theorem:

$$\lim_{N \to \infty} \frac{N D_N(n\alpha)}{\log N \log\log N} = \frac{2}{\pi^2} \qquad \text{in measure.}$$

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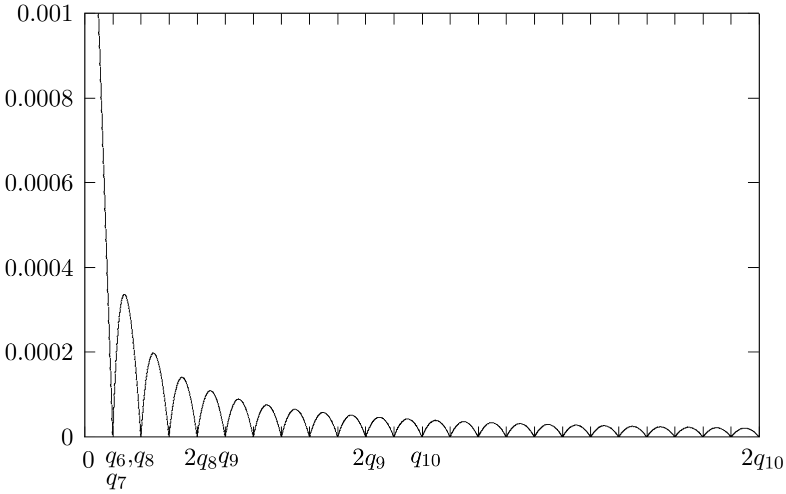

FIGURE 1. $D^*_N(n\alpha)$, $\alpha = 1 - \log_{10} 7$, up to $N = 2q_{10}$, every 10,000 points. q_6 = 2,455,069, q_7 = 2,455,579, q_8 = 4,910,648, $2q_8$ = 9,821,296, q_9 = 12,276,875, $2q_9$ = 24,553,750, q_{10} = 29,464,398, $2q_{10}$ = 58,928,796.

Kesten's result means that in a stochastic sense, $D_N(n\alpha)$ decreases regularly, with a speed of $cN^{-1} \log N \log\log N$. However, this does not mean a regular behavior of $D_N(n\alpha)$ for almost all α. Setokuchi and Takashima [7] showed that isolated large digits in the continued fraction expansion of α lead to a series of inverted parabolas in the graph of $D_N(n\alpha)$ and the purpose of the present paper is to investigate this phenomenon in more detail, proving that the effect of a large digit is much longer than shown in [7].

Setokuchi and Takashima [7] gave very detailed estimations for the discrepancies by improving Schoissengeier's results in [5], and they report the unusual behavior of the discrepancies of irrational rotations based on some specific numbers having single isolated large partial quotient a_η (η denotes the order of isolated large partial quotient) in their continued fraction expansion. They reveal how the existence of a single isolated large partial quotient does influence the behavior of the discrepancies. They, however, consider N's only over rather short range (at most 10 millions), and they explain 4 or 5 repetitions of "hills". Thus it seems insufficient to compare their results and the limiting behavior shown by Kesten [2].

In this paper we extend estimates in [7], and give some general conditions for repetitions of hills of discrepancies. In improving and extending their results, we take into account the fact that there are two types of "valleys", one-point valley and wider valley. We give how one-point valleys and wider valleys will emerge according to whether a_j is equal to or is larger than 1. We improve some theorems in [7] by using information derived from such considerations on emergence of valleys. We apply our results on much larger N's and explain repetitions of much more hills than those dealt with in [7]. For example, in case of $1 - \log_{10} 7$, our general conditions are valid over an incomparably longer range of N, discussed in [7]. We show that hills repeat unexpectedly many times, more than 2.7×10^{27} times, where N runs up to over 6.8×10^{33}. For other irrational numbers with single isolated large partial quotient which are not restricted to the common logarithm, our general conditions ensure occurrences of repeated hills much more times. Schoissengeier's results are restricted to "finite" ranges of N's, and our results are also restricted to finite ranges. Although our results are, of course, not directly comparable with Kesten's limit theorem, they can be applied for long ranges of N's.

We consider, as examples, not only the case of $1 - \log_{10} 7$, but also the cases of $2 - \log_{10} 33$ and $2 - \log_{10} 54$. Our results are valid for general irrational numbers with single isolated large partial quotient, and these specific irrational numbers play very interesting and important role in the problem of the leading digits of the power a^n, for $a = 33$ or $a = 54$. The problem of the leading digits of a^n is closely related to the studies of irrational rotations based on $\log_{10} a$. Mori and Takashima [4] reported very unusual phenomena observed in the behavior of the chi-square tests of the leading digits of a^n, for $a = 7$, $a = 33$ and $a = 54$. For these natural numbers, we will show later that the continued fraction expansions of $\log_{10} a$ have isolated large partial quotients. Mori and Takashima [4] gave mathematical explanations for such unusual phenomena, and they show the unusual graphs in these cases with slightly different shapes. These problems of the leading digits are very deeply connected with the problems of the discrepancies, and we have much interest in the discrepancies of irrational rotations based on such specific numbers.

2. Repetitions of parabola-like hills

We first provide some notions and notations according to Drmota and Tichy [1]. Let $(n\alpha)_{n\in\mathbb{N}}$ be an irrational rotation based on an irrational number α, $0 < \alpha < 1/2$,

$$(n\alpha)_{n\in\mathbb{N}} = \{\{n\alpha\} : n \in \mathbb{N}\},$$

and we use discrepancies D_N^* of $(n\alpha)_{n\in\mathbb{N}}$ defined by

$$D_N^*(n\alpha) = \sup_{0<a\le 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[0,a)}(\{n\alpha\}) - a \right|.$$

It is well-known that always $D_N^* \le D_N \le 2D_N^*$ and two types of discrepancies D_N and D_N^* have essentially the same behavior. In this paper we consider only D_N^*. We denote the continued fraction expansion of α by

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0; a_1, a_2, \ldots]$$

with convergents $r_n = p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$. For a natural number N, the Ostrowski representation of N to base α is given as follows:

$$N = \sum_{j=0}^{m} b_j q_j$$

with $0 \le b_j \le a_{j+1}$ for $0 < j < m$, $0 \le b_0 < a_1$, $0 < b_m \le a_{m+1}$, and $b_{j-1} = 0$ if $b_j = a_{j+1}$, where $q_m \le N < q_{m+1}$.

Since emergence of valleys and hills depend on α, we first consider mainly the case where $\alpha = 1 - \log_{10} 7$. For other α having single isolated large partial quotient, our arguments are still valid with appropriate modifications. The continued fraction expansion of $1 - \log_{10} 7$ is given by

$$\begin{aligned} 1 - \log_{10} 7 = [0;&6, 2, 5, 6, 1, 4813, 1, 1, 2, 2, 2, 1, 1, 1, 6, 5, 1, 83, 7, 2,\\ &1, 1, 1, 8, 5, 21, 1, 1, 3, 2, 1, 4, 2, 3, 14, 2, 6, 1, 1, 5,\\ &2, 1, 2, 4, 26, 2, 6, 1, 5, 1, 1, 2, 2, 3, 6, 2, 2, 103, 2, 2,\\ &1084, \ldots], \end{aligned} \tag{1}$$

and $1 - \log_{10} 7$ has a single isolated large partial quotient $a_6 = 4813$ in the early part of its continued fraction expansion. We then denote the order 6 of a_6 by η, that is, $\eta = 6$. This η may vary according to α.

2.1. **Upper estimates of endpoints of valleys**

Table 1 shows endpoints of valleys in Fig. 1 for q_η $(= q_6) \le N \le q_{10}$. The valleys for $q_{10} < N \le 2q_{10}$ are similar to Table 1 by adding q_{10}. Then there are two types of valleys in Fig. 1, "one-point valley" and "wider valley":

2nd, 4th, 7th, 9th, 12th valleys, and so on, consist of only one-point N, whereas 1st, 3rd, 5th, 6th, 8th, 10th, 11th valleys, and so on, have two endpoints, for example, 1st valley has endpoints q_η and $q_{\eta+1}$ (see Fig. 2). These two-point valleys are a little "wider" than the one-point valleys. They have a common "width" $q_{\eta-1} = q_{\eta+1} - q_\eta$ $(= 510)$.

In general, endpoints of valleys are represented by the following specific values of N, having coefficients $b_0 = \cdots = b_{\eta-1} = 0$ and $0 \le b_j \le a_{j+1}$ for

Table 1. Endpoints of the kth valley in Fig. 1, $k = 1, 2, \dots, 12$.

k	1	2	3	4	5	6	7
N	q_6 q_7	q_8	$q_8 + q_6$ $q_8 + q_7$	$2q_8$	$2q_8 + q_6$ q_9 $(= 2q_8 + q_7)$	$q_9 + q_6$ $q_9 + q_7$	$q_9 + q_8$

k	8	9	10	11	12
N	$q_9 + q_8 + q_6$ $q_9 + q_8 + q_7$	$q_9 + 2q_8$	$q_9 + 2q_8 + q_6$ $2q_9$ $(= q_9 + 2q_8 + q_7)$	$2q_9 + q_6$ $2q_9 + q_7$	q_{10}

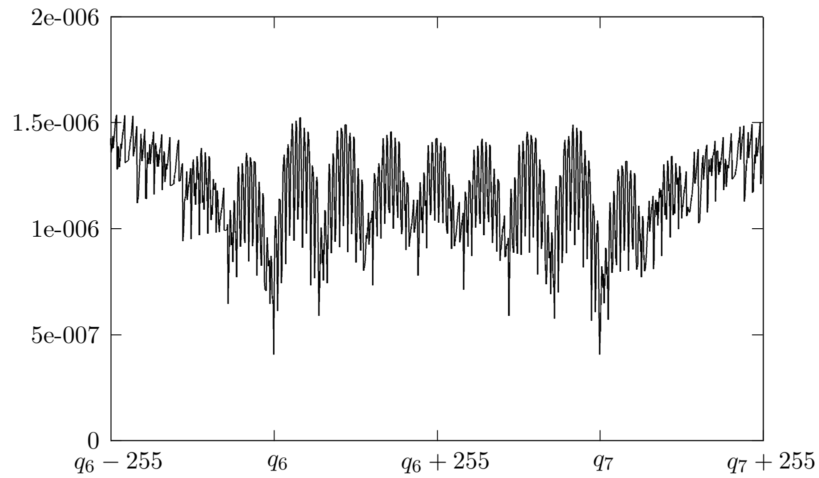

Figure 2. $D_N^*(n\alpha)$, $\alpha = 1 - \log_{10} 7$, $N \in [q_6 - 255, q_7 + 255)$, every point. $q_6 = 2{,}455{,}069$, $q_7 = 2{,}455{,}579$.

$\eta \le j \le m$ with $b_m > 0$:

$$N = \sum_{j=\eta}^{m} b_j q_j. \tag{2}$$

Setokuchi and Takashima [7] treated mainly 1st, 2nd, 3rd and 4th valleys (cf. [7, Theorem 4.2]). We extend their estimates to obtain estimates for more general valleys, for example, 5th, 6th, 7th valleys, and so on. We first give the following result:

Theorem 2.1. *Let $0 < \alpha < 1/2$ be an irrational number. Put $i_N = \min\{j \geq 0 : b_j \neq 0\}$. Then we have*

$$ND_N^*(n\alpha) < \max\left(\sum_{j=[(i_N+1)/2]}^{[m/2]} b_{2j}, \sum_{j=[i_N/2]}^{[(m-1)/2]} b_{2j+1}\right) + \left\lceil\frac{m-i_N}{2}\right\rceil + 2 \quad (3)$$

for $N = \sum_{j=\eta}^{m} b_j q_j$, $0 \leq b_j \leq a_{j+1}$ for $\eta \leq j \leq m$ with $b_m > 0$.

Let

$$M(N) = \max\left(\sum_{\substack{j=0 \\ j\neq(\eta-1)/2}}^{[m/2]} a_{2j+1}, \sum_{\substack{j=1 \\ j\neq\eta/2}}^{[(m+1)/2]} a_{2j}\right). \quad (4)$$

From Theorem 2.1, we obtain the following estimate, which is simpler than that of Theorem 2.1.

Theorem 2.2. *Let $0 < \alpha < 1/2$ be an irrational number. If $\eta \geq 3$, then we have*

$$D_N^*(n\alpha) < \frac{2M(N)}{N} \quad (5)$$

for $N = \sum_{j=\eta}^{m} b_j q_j$, $0 \leq b_j \leq a_{j+1}$ for $\eta \leq j \leq m$ with $b_m > 0$.

Theorems 2.1 and 2.2 give upper estimates for general endpoints of valleys. For example, in case of $1 - \log_{10} 7$, these theorems are applicable not only to endpoints of valleys in Fig. 1, but also to endpoints of valleys beyond the range shown in Fig. 1. We use Theorem 2.2 later in Subsection 2.3 together with Theorems 2.3 and 2.4 to compare the magnitude of $D_N^*(n\alpha)$ for peaks of hills with the magnitude of $D_N^*(n\alpha)$ for valleys.

2.2. **Upper estimates over wider valleys**

We next give estimates for the whole of a wider valley. In [7], Setokuchi and Takashima gave the upper estimate for wider valleys (cf. [7, Theorem 4.3]). Their estimate is, however, only for 1st and 3rd valleys.

To give estimates for more general wider valleys, we introduce the following notation:

$$I_w = \bigcup_{k=\eta+1}^{m}\left(\bigcup_{b_j' s}\left[\sum_{\substack{j=\eta+1 \\ b_{\eta+1}\neq a_{\eta+2}}}^{k} b_j q_j + a_{\eta+1} q_\eta, \sum_{\substack{j=\eta+1 \\ b_{\eta+1}\neq a_{\eta+2}}}^{k} b_j q_j + q_{\eta+1}\right)\right), \quad (6)$$

where $\cup_{b_j's}$ denotes the union of intervals over b_j's, satisfying the Ostrowski representation rule; $0 \leq b_j \leq a_{j+1}$ for $\eta+1 \leq j \leq k$, and $b_{j-1} = 0$ if $b_j = a_{j+1}$.

In the notation (6), each $[\cdot)$ denotes one wider valley, and I_w denotes the union of all wider valleys up to q_{m+1}. The length of each wider valley is equal to $q_{\eta-1} = q_{\eta+1} - a_{\eta+1}q_\eta$, by the well-known recurrence formula $q_{j+1} = a_{j+1}q_j + q_{j-1}$.

Remark 1. The problem how wider valleys occur depends on $a_{\eta+1}$, $a_{\eta+2}$, …, a_m. For example, in the interval $[q_\eta, q_{\eta+1})$, one wider valley formed by

$$[a_{\eta+1}q_\eta, q_{\eta+1})$$

occurs, and points $N = b_\eta q_\eta$, $1 \leq b_\eta < a_{\eta+1}$, are one-point valley. On the other hand, in the interval $[q_{\eta+1}, q_{\eta+2})$, wider valleys formed by

$$[b_{\eta+1}q_{\eta+1} + a_{\eta+1}q_\eta, (b_{\eta+1}+1)q_{\eta+1}),$$

$1 \leq b_{\eta+1} < a_{\eta+2}$, occur $(a_{\eta+2}-1)$ times, and points $N = b_{\eta+1}q_{\eta+1} + b_\eta q_\eta$, $1 \leq b_{\eta+1} < a_{\eta+2}$, $1 \leq b_\eta < a_{\eta+1}$, are one-point valley. For further intervals, the problem of occurrence of wider valleys becomes somewhat more complicated. In the interval $[q_{\eta+2}, q_{\eta+3})$, wider valleys occur $(a_{\eta+3} + (a_{\eta+3}-1)(a_{\eta+2}-1))$ times; in the interval $[q_{\eta+3}, q_{\eta+4})$, wider valleys occur $(a_{\eta+4} + a_{\eta+4}(a_{\eta+2}-1) + (a_{\eta+4}-1)a_{\eta+3} + (a_{\eta+4}-1)(a_{\eta+3}-1)(a_{\eta+2}-1))$ times; and so on.

We then obtain the following upper estimate for $D_N^*(n\alpha)$ for much more wider valleys than those dealt with in [7].

Theorem 2.3. *Let $0 < \alpha < 1/2$ be an irrational number. If $M(N)$ satisfies*

$$M(N) \geq \left[\frac{m+1}{2}\right] + 2, \tag{7}$$

then we have

$$D_N^*(n\alpha) < \frac{2M(N)}{N} \tag{8}$$

for $N \in I_w$.

By the definition (4) of $M(N)$, we note that the condition (7) in Theorem 2.3 is not too strict: if α satisfies $M(N) > [(m+1)/2] + 2$ for some $N' \geq q_\eta$, then the condition (7) is satisfied for all $N \geq N'$. For example, in case of $\alpha = 1 - \log_{10} 7$, the condition (7) is satisfied for all $N \geq q_\eta$. Therefore Theorem 2.3 is still valid for N ($\in I_w$) greater than the points treated in Fig. 1.

Remark 2. The upper bound of Theorem 2.3, (8), is the same as that of Theorem 2.2, (5). Theorem 2.2 might seem to be included Theorem 2.3, at first sight, but $N = \sum_{j=\eta}^{m} b_j q_j$ represented by (2), does not always belong to I_w. The coefficient b_η in (2) takes values from 0 to $a_{\eta+1}$, and only in the case when $b_\eta = a_{\eta+1}$, N belongs to I_w.

2.3. **Estimates for parabola-like hills**

Next we discuss repetitions of parabola-like hills. Estimates for $ND_N^*(n\alpha)$ for hills given in [7] are concerned only with several hills (cf. [7, Theorems 4.4 and 4.5]). We can, here, give the following estimate for $ND_N^*(n\alpha)$ for much more hills.

Theorem 2.4. *Let $0 < \alpha < 1/2$ be an irrational number. Assume that α has single isolated large partial quotient a_η satisfying*

$$a_\eta > 12M(N). \tag{9}$$

If $M(N)$ satisfies

$$M(N) \geq 5, \tag{10}$$

then we have

$$-M(N) < ND_N^*(n\alpha) - b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) < 3M(N) \tag{11}$$

for $N \notin I_w$.

Theorem 2.4 ensures that parabola-like hills repeat while the condition (9) is satisfied: let $f(x)$ be the quadratic function, $f(x) = x(1-x)$, $0 \leq x \leq 1$. Then $f(0) = f(1) = 0$ and $f(1/2) = 1/4$. Then the estimate in Theorem 2.4, (11), can be rewritten in the form

$$-M(N) < ND_N^*(n\alpha) - a_\eta f(x') < 3M(N) \quad \text{for } x' = \frac{N'}{a_\eta q_{\eta-1}},$$

where $N' = N - \sum_{j=0, j\neq\eta-1}^{m} b_j q_j$ if $N \notin I_w$. It is easily seen that the "peak" of each "hill" is estimated from below by $a_\eta f(1/2) - M(N)$. Thus, for α satisfying the condition (9), $a_\eta > 12M(N)$, the peak of each hill is much larger than the upper bound $2M(N)$ at the valley's shown in Theorem 2.2, (5), and Theorem 2.3, (8),

$$a_\eta f\left(\frac{1}{2}\right) - M(N) \fallingdotseq \frac{a_\eta}{4} - M(N) > 2M(N).$$

By using Theorem 2.4, we consider how many times parabola-like hills repeat.

Example 1. In case of $\alpha = 1 - \log_{10} 7$, α satisfies the condition (9) for $m \leq 59$ ($N < q_{60}$), but does not satisfy the condition (9) for $m \geq 60$ ($N \geq q_{60}$) due to the influence of rather large 61st partial quotient, $a_{61} = 1084$:

$$M(N) = \max\left(\sum_{\substack{j=1 \\ j\,\mathrm{odd},\, j\neq\eta}}^{59} a_j, \sum_{\substack{j=2 \\ j\,\mathrm{even},\, j\neq\eta}}^{60} a_j \right) = 272, \quad \text{if } m = 59,$$

$$M(N) = \max\left(\sum_{\substack{j=1 \\ j\,\mathrm{odd},\, j\neq\eta}}^{61} a_j, \sum_{\substack{j=2 \\ j\,\mathrm{even},\, j\neq\eta}}^{60} a_j \right) = 1205, \quad \text{if } m = 60.$$

Thus Theorem 2.4 ensures that hills occur repeatedly while N runs up to

$$q_{60} = 6{,}856{,}445{,}927{,}562{,}934{,}919{,}532{,}922{,}886{,}126{,}232 \fallingdotseq 6.8 \times 10^{33}$$

while the period between valleys equals $q_\eta = 2455069$. This implies that hills repeat more than 2.7×10^{27} times,

$$\frac{q_{60}}{q_\eta + q_{\eta-1}} > 2.7 \times 10^{27},$$

in consideration of the width of wider valleys $q_{\eta-1} = 510$. The above explanation suggests that hills do not vanish under the influence of slightly large partial quotients $a_{18} = 83$ and $a_{58} = 103$ (cf. (1)). To calculate the values of $D_N^*(n\alpha)$ for such unusually large N's is too heavy work for usual computers. Theorem 2.4 ensures, however, repetitions of hills theoretically up to $N = 6.8 \times 10^{33}$.

In the following of this subsection we give other examples, $2 - \log_{10} 33$ and $2 - \log_{10} 54$.

Example 2. The continued fraction expansion of $2 - \log_{10} 33$ is given by

$$2 - \log_{10} 33 = [0; 2, 13, 299, 1, 1, 10, 1, 14, \ldots],$$

and $2 - \log_{10} 33$ has a rather large isolated 3rd partial quotient, $a_3 = 299$ (i.e., $\eta = 3$). In this case, the condition (9) holds for $m \leq 6$ ($N < q_7$), but it does not hold for $m \geq 7$ ($N \geq q_7$), because

$$M(N) = 24, \quad \text{if } m = 6,$$
$$M(N) = 38, \quad \text{if } m = 7.$$

Thus hills repeat 22 times during $N \in [q_\eta, q_7)$ (see Fig. 3). Since the large partial quotient 299 is not too large in comparison with other quotients $a_2 = 13$, $a_6 = 10$ and $a_8 = 14$, the condition (9) fails for relatively small m.

Example 3. The continued fraction expansion of $2 - \log_{10} 54$ is given by

$$2 - \log_{10} 54 = [0; 3, 1, 2, 1, 3, 1, 326, 2, 1, 3, 1, 5, 4, 1, 1, 1, 2, 26, \ldots],$$

and $2 - \log_{10} 54$ has a rather large isolated 7th partial quotient, $a_7 = 326$ (i.e., $\eta = 7$). In this case, the condition (9) holds for $m \leq 16$ ($N < q_{17}$), but it does not hold for $m \geq 17$ ($N \geq q_{17}$) due to the influence of the quotient $a_{18} = 26$:

$$M(N) = 17, \quad \text{if } m = 16,$$
$$M(N) = 41, \quad \text{if } m = 17.$$

Thus hills starting with $N = q_\eta$ in Fig. 4 repeat more than 3102 times while N runs up to $q_{17} = 72214041$,

$$\frac{q_{17}}{q_\eta + q_{\eta-1}} > 3102,$$

where $q_\eta = 23202$ and $q_{\eta-1} = 71$. The large partial quotient 326 is almost as large as the quotient 299 in case of $2 - \log_{10} 33$. The condition (9) is, however, satisfied for a more wider range of N compared with the case of $2 - \log_{10} 33$, because, in case of $2 - \log_{10} 54$, the quotients up to a_{17} except the quotient 326 are very small.

Remark 3. In the cases $\alpha = 2 - \log_{10} 33$ and $\alpha = 2 - \log_{10} 54$, the condition (9) fails for $N = q_7$ and $N = q_{17}$, respectively. We can, however, observe that hills still occur on the range beyond above N's (see Figs. 5 and 6).

3. Proofs for results of Section 2

Before giving proofs, we state some results given in [1, 5, 6, 7] that we need to prove theorems. Let us define the numbers A_j by

$$A_j = N_{j-1}(\alpha - r_j) + \sum_{t=j}^{m} b_t(q_t\alpha - p_t)$$

for $0 \leq j \leq m + 2$ and $A_{-1} = 0$, where $N_j = \sum_{t=0}^{j} b_t q_t$ for $0 \leq j \leq m$, $N_{-1} = 0$ and $N_m = N_{m+1} = N$. Then A_j satisfies the following properties:

Lemma 3.1. ([1, Lemma 1.62], [7, Section 2, Equation (4)], [5, Section 3, Proposition 1])

(i) *If $b_j \neq 0$, then $(-1)^j A_j > 0$ for $0 \leq j \leq m$.*
(ii) *Let $P = \{j : A_j > 0, 0 \leq j \leq m\}$. If $j \notin P$ is even, or if $j \in P$ is odd, then $a_{j+1}q_jA_j = q_{j+1}A_{j+1} - q_{j-1}A_{j-1}$ for $0 \leq j \leq m$.*
(iii) *For $0 \leq j \leq m$, we have $-1/q_{j+1} < (-1)^j A_j < 1/q_j$.*

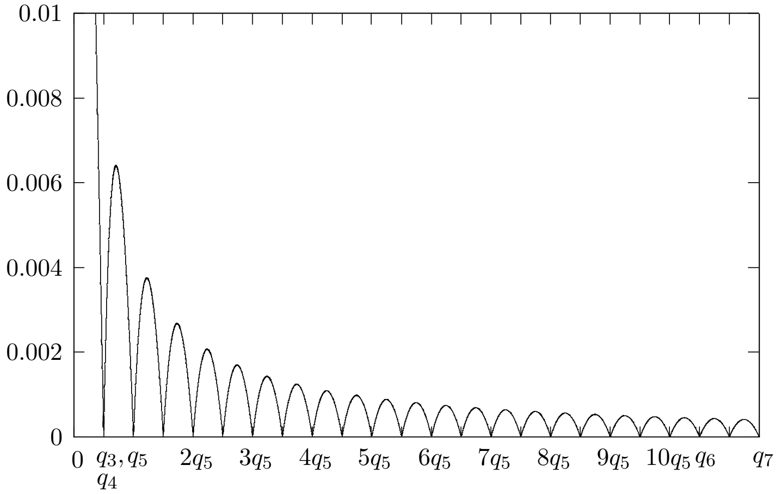

FIGURE 3. $D_N^*(n\alpha)$, $\alpha = 2 - \log_{10} 33$, up to $N = q_7$, every 10 points. $q_3 = 8{,}075$, $q_4 = 8{,}102$, $q_5 = 16{,}177$, $q_6 = 169{,}872$, $q_7 = 186{,}049$.

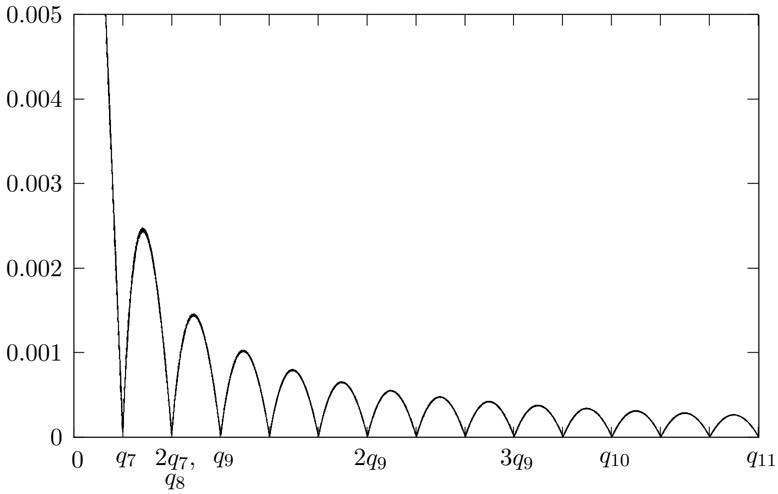

FIGURE 4. $D_N^*(n\alpha)$, $\alpha = 2 - \log_{10} 54$, up to $N = q_{11}$, every 10 points. $q_7 = 23{,}202$, $q_8 = 46{,}475$, $q_9 = 69{,}677$, $q_{10} = 255{,}506$, $q_{11} = 325{,}183$.

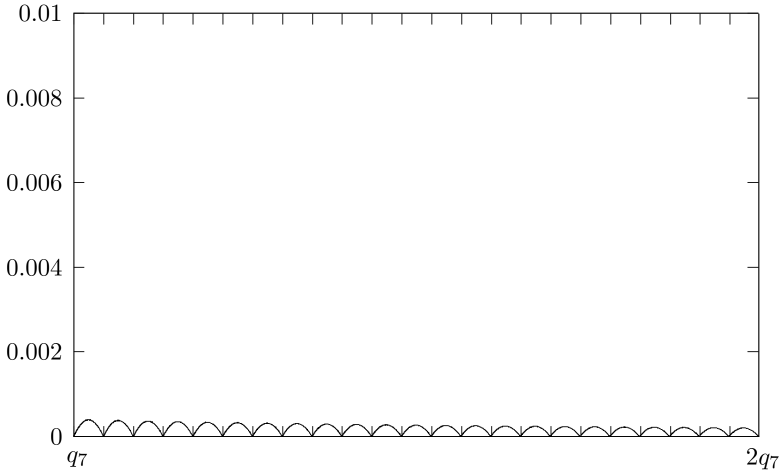

FIGURE 5. $D_N^*(n\alpha)$, $\alpha = 2 - \log_{10} 33$, $N \in [q_7, 2q_7)$, every 10 points. $q_7 = 186{,}049$, $2q_7 = 372{,}098$.

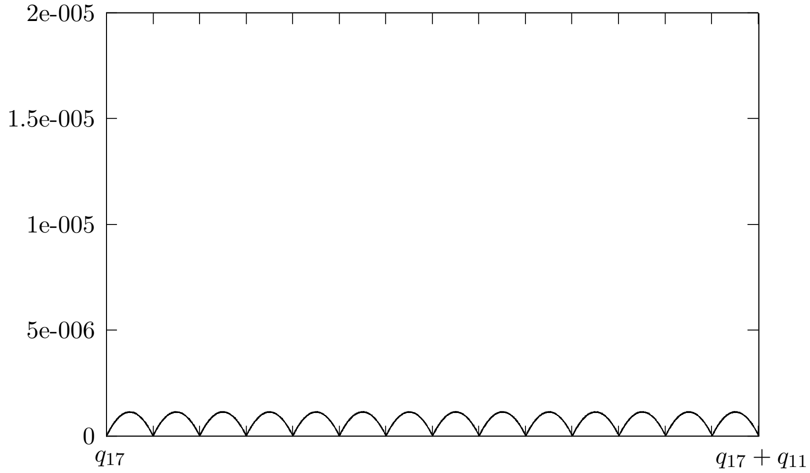

FIGURE 6. $D_N^*(n\alpha)$, $\alpha = 2 - \log_{10} 54$, $N \in [q_{17}, q_{17} + q_{11})$, every 10 points. $q_{11} = 325{,}183$, $q_{17} = 72{,}214{,}041$, $q_{17} + q_{11} = 72{,}539{,}224$.

The proof of Theorem 2.1 is based on the following Schoissengeier's result for $ND_N^*(n\alpha)$ given in the proof of Corollary 1 of Theorem 1 in [6]. We refer to this result as a separate theorem for convenient reference.

Theorem A. ([6, p.56]) *Let $0 < \alpha < 1/2$ be an irrational number. Then*

$$ND_N^*(n\alpha) = \sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) + \sum_{\substack{j \text{ odd} \\ j \in P}} a_{j+1}q_jA_j - \sum_{\substack{j \text{ even} \\ j \notin P}} a_{j+1}q_jA_j + \max\left(0, A_0 - \sum_{j=0}^{m} b_j((-1)^j - q_jA_j)\right) + \varepsilon, \tag{12}$$

where the error term ε satisfies $|\varepsilon| \le 1$, and $P = \{j : A_j > 0,\ 0 \le j \le m\}$.

The proofs of Theorems 2.3 and 2.4, on the other hand, are based on the following theorem, which is a refinement of Theorem A, due to Setokuchi and Takashima [7].

Theorem 3.2. ([7, Theorem 3.1]) *Let $0 < \alpha < 1/2$ be an irrational number. Then*

$$ND_N^*(n\alpha) = \max\left(\sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}),\ \sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1 + q_{2j+1}A_{2j+1}) + A_0\right) + c, \tag{13}$$

where the error term c satisfies $-1 \le c < [(m+1)/2] + 1$.

To estimate the terms $b_{2j}(1-q_{2j}A_{2j})$ and $b_{2j+1}(1+q_{2j+1}A_{2j+1})$ in Theorem A, (12), and Theorem 3.2, (13), we use the following two lemmas:

Lemma 3.3. ([7, Lemma 5.1]) *Let α be a positive irrational number. Then for $0 \le j \le [m/2]$ we have*

$$0 \le b_{2j}(1 - q_{2j}A_{2j}) \le b_{2j}, \tag{14}$$

with equality if and only if $b_{2j} = 0$. Moreover, for $0 \le j \le [(m-1)/2]$ we have

$$0 \le b_{2j+1}(1 + q_{2j+1}A_{2j+1}) \le b_{2j+1}, \tag{15}$$

with equality if and only if $b_{2j+1} = 0$.

Lemma 3.4. ([7, Lemma 5.2]) *Let α be a positive irrational number. Then for $0 \le j \le [m/2]$ we have*

$$b_{2j}\left(1 - \frac{b_{2j}}{a_{2j+1}}\right) - 2 < b_{2j}(1 - q_{2j}A_{2j}) < b_{2j}\left(1 - \frac{b_{2j}}{a_{2j+1}}\right) + 3. \tag{16}$$

Moreover, for $0 \le j \le [(m-1)/2]$ *we have*

$$b_{2j+1}\left(1-\frac{b_{2j+1}}{a_{2j+2}}\right)-2 < b_{2j+1}(1+q_{2j+1}A_{2j+1}) < b_{2j+1}\left(1-\frac{b_{2j+1}}{a_{2j+2}}\right)+3. \tag{17}$$

Since a_η is large, the coefficient $b_{\eta-1}$ can vary very widely, $0 \le b_{\eta-1} \le a_\eta$. Then the degree of accuracy of the estimates (16) (resp. (17)) in Lemma 3.4 is considerably better than that of the estimate (14) (resp. (15)) in Lemma 3.3. Thus we use Lemma 3.4, (16) (resp. (17)), only for the specific term $b_{\eta-1}(1-q_{\eta-1}A_{\eta-1})$ (resp. $b_{\eta-1}(1+q_{\eta-1}A_{\eta-1})$).

We first prove Theorem 2.1. The proof is based on Theorem A.

Proof of Theorem 2.1. We prove only the case where

$$i_N = \min\{j \ge 0 : b_j \ne 0\}$$

is even. The proof of the case where i_N is odd can be shown by similar arguments. Assume that N has coefficients $b_0 = \cdots = b_{\eta-1} = 0$ and $0 \le b_j \le a_{j+1}$ for $\eta \le j \le m$,

$$N = \sum_{j=\eta}^{m} b_j q_j.$$

Then we clearly have $A_0 = \cdots = A_{i_N} = \sum_{t=i_N}^{m} b_t(q_t\alpha - p_t)$ because $b_0 = \cdots = b_{i_N-1} = 0$. Thus, from $b_{i_N} \ne 0$ and Lemma 3.1 (i), we have $A_j > 0$ for $0 \le j \le i_N$. This implies that $\{j : j \in P \text{ is odd},\ 1 \le j \le i_N - 1\} = \{1, 3, \ldots, i_N - 1\}$, and consequently

$$\sum_{\substack{j\text{ odd}\\ j\in P}} a_{j+1}q_jA_j = q_{i_N}A_{i_N} - A_0 + \sum_{\substack{j=i_N+1\\ j\text{ odd}\\ j\in P}}^{m} a_{j+1}q_jA_j, \tag{18}$$

where we used Lemma 3.1 (ii) for $1 \le j \le i_N - 1$. On the other hand, since $\{j : j \notin P \text{ is even},\ 0 \le j \le i_N\} = \emptyset$, we have

$$\sum_{\substack{j\text{ even}\\ j\notin P}} a_{j+1}q_jA_j = \sum_{\substack{j=i_N+2\\ j\text{ even}\\ j\notin P}}^{m} a_{j+1}q_jA_j. \tag{19}$$

We now show that

$$\sum_{\substack{j\text{ odd}\\ j\in P}} a_{j+1}q_jA_j - \sum_{\substack{j\text{ even}\\ j\notin P}} a_{j+1}q_jA_j < q_{i_N}A_{i_N} - A_0 + \left[\frac{m-i_N}{2}\right]. \tag{20}$$

If $j \notin P$ is even, then $j-1 \notin P$ and $j+1 \notin P$; on the other hand, if $j \in P$ is odd, then $j-1 \in P$ and $j+1 \in P$ (cf. [6, p.55]). Thus it follows from $A_{i_N} > 0$ and $(-1)^m A_m > 0$ that

$$\begin{aligned} 0 \le \sharp\{j : j \in P \text{ is odd},\ i_N+1 \le j \le m\} \\ + \sharp\{j : j \notin P \text{ is even},\ i_N+2 \le j \le m\} \le [(m-i_N)/2], \end{aligned} \tag{21}$$

where $\sharp(A)$ denotes the number of elements of a set A. Using (21) and the facts that $a_{j+1}q_jA_j < 1$ if $j \in P$ is odd, and $-a_{j+1}q_jA_j < 1$ if $j \notin P$ is even (cf. [7, Proof of Theorem 3.1]), we have

$$\sum_{\substack{j=i_N+1 \\ j \text{ odd} \\ j \in P}}^{m} a_{j+1}q_jA_j - \sum_{\substack{j=i_N+2 \\ j \text{ even} \\ j \notin P}}^{m} a_{j+1}q_jA_j < \left[\frac{m-i_N}{2}\right].$$

This, together with (18) and (19), we arrive at (20). Applying the estimate (20) and the relation $b_0 = \cdots = b_{i_N-1} = 0$ to Theorem A, (12), we obtain

$$\begin{aligned} &ND_N^*(n\alpha) \\ &< \max\left(\sum_{j=[(i_N+1)/2]}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}), \sum_{j=[i_N/2]}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) + A_0\right) \\ &\quad + q_{i_N}A_{i_N} - A_0 + \left[\frac{m-i_N}{2}\right] + \varepsilon, \end{aligned}$$

where $|\varepsilon| \le 1$. Since the specific coefficient $b_{\eta-1}$ is always 0 (recall that $i_N \ge \eta$), we use only Lemma 3.3 to estimate the terms $b_{2j}(1-q_{2j}A_{2j})$ and $b_{2j+1}(1+q_{2j+1}A_{2j+1})$ in the above expression. As a result, we obtain

$$ND_N^*(n\alpha) < \max\left(\sum_{j=[(i_N+1)/2]}^{[m/2]} b_{2j}, \sum_{j=[i_N/2]}^{[(m-1)/2]} b_{2j+1}\right) + \left[\frac{m-i_N}{2}\right] + 2,$$

where we used $q_{i_N}A_{i_N} < 1$. □

We next derive Theorem 2.2 from Theorem 2.1.

Proof of Theorem 2.2. As for the maximum term on the right-hand side of (3) in Theorem 2.1, we have

$$\sum_{j=[(i_N+1)/2]}^{[m/2]} b_{2j} = \sum_{\substack{j=0 \\ j \ne (\eta-1)/2}}^{[m/2]} b_{2j} < \sum_{\substack{j=0 \\ j \ne (\eta-1)/2}}^{[m/2]} a_{2j+1},$$

$$\sum_{j=[i_N/2]}^{[(m-1)/2]} b_{2j+1} = \sum_{\substack{j=0 \\ j\neq(\eta-2)/2}}^{[(m-1)/2]} b_{2j+1} < \sum_{\substack{j=1 \\ j\neq\eta/2}}^{[(m+1)/2]} a_{2j},$$

and consequently the maximum term in Theorem 2.1 is less than $M(N)$. Thus it follows from Theorem 2.1 that

$$ND_N^*(n\alpha) < M(N) + \left[\frac{m-i_N}{2}\right] + 2. \tag{22}$$

By the property $i_N \geq \eta$ and the assumption $\eta \geq 3$, we have $[(m-i_N)/2]+2 \leq [(m+1)/2]$. Moreover, we have $[(m+1)/2] \leq M(N)$ for general irrational numbers, because $a_j > 0$ for $1 \leq j \leq m+1$. Thus we obtain $[(m-i_N)/2]+2 \leq M(N)$. This and (22) yield the upper bound $2M(N)$. □

In the latter part of this section, we give the proofs of Theorems 2.3 and 2.4. The main tools of the proofs are Theorem 3.2, Lemmas 3.3 and 3.4, which are valid for large but not infinite N's. Setokuchi and Takashima [7] gave the proofs of the upper estimates of wider valleys and hills only for $\alpha = 1 - \log_{10} 7$ and only over rather short ranges of N's. We, however, give the proofs for irrational α's having single isolated large a_η and for long ranges of N's.

Proof of Theorem 2.3. Assume that $N \in I_w$. Then the specific coefficient $b_{\eta-1}$ is always 0 because $b_\eta = a_{\eta+1}$ (recall that $b_{j-1} = 0$ if $b_j = a_{j+1}$). Thus we use only Lemma 3.3 to estimate the terms $b_{2j}(1-q_{2j}A_{2j})$ and $b_{2j+1}(1+q_{2j+1}A_{2j+1})$ in Theorem 3.2, (13), and we have

$$\sum_{j=0}^{[m/2]} b_{2j}(1-q_{2j}A_{2j}) < \sum_{\substack{j=0 \\ j\neq(\eta-1)/2}}^{[m/2]} a_{2j+1},$$

$$\sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1+q_{2j+1}A_{2j+1}) < \sum_{\substack{j=1 \\ j\neq\eta/2}}^{[(m+1)/2]} a_{2j}.$$

Applying these estimates to Theorem 3.2, (13), we obtain

$$ND_N^*(n\alpha) < M(N) + |A_0| + c < M(N) + \left[\frac{m+1}{2}\right] + 2,$$

where we used $|A_0| < 1$ (cf. Lemma 3.1 (iii)) and $c < [(m+1)/2]+1$. Thus, from the condition (7), $M(N) \geq [(m+1)/2]+2$, we arrive at the upper bound $2M(N)$. □

Proof of Theorem 2.4. Let I denote an interval such that the condition (9), $a_\eta > 12M(N)$, is satisfied. Assume that $N \in I \setminus I_w$. Then the specific coefficient $b_{\eta-1}$ takes values $0 \le b_{\eta-1} \le a_\eta$ over each interval with the length $q_{\eta-1}$ included in $I \setminus I_w$. We prove only the case where η is even. The proof of the case where η is odd, can be shown by similar arguments. By Lemma 3.3, (14), we can estimate the first sum in Theorem 3.2, (13), as follows:

$$0 \le \sum_{j=0}^{[m/2]} b_{2j}(1 - q_{2j}A_{2j}) < \sum_{\substack{j=0 \\ j\ne(\eta-1)/2}}^{[m/2]} a_{2j+1}.$$

On the other hand, we use Lemma 3.4, (17), for $b_{\eta-1}(1 + q_{\eta-1}A_{\eta-1})$, and we apply Lemma 3.3, (15), to the other terms $b_{2j+1}(1 + q_{2j+1}A_{2j+1})$. As a result, we have the following estimate for the second sum in Theorem 3.2, (13):

$$\begin{aligned} b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) - 2 &< \sum_{j=0}^{[(m-1)/2]} b_{2j+1}(1 + q_{2j+1}A_{2j+1}) \\ &< \sum_{\substack{j=1 \\ j\ne\eta/2}}^{[(m+1)/2]} a_{2j} + b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 3. \end{aligned}$$

By combining these two estimates with Theorem 3.2, (13), we obtain

$$\begin{aligned} & b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) - 2 - |A_0| + c \\ & < ND_N^*(n\alpha) < b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) + 3 + M(N) + |A_0| + c. \end{aligned}$$

Since $|A_0| < 1$ (cf. Lemma 3.1 (iii)) and $-1 \le c < [(m+1)/2] + 1$, we have

$$-4 < ND_N^*(n\alpha) - b_{\eta-1}\left(1 - \frac{b_{\eta-1}}{a_\eta}\right) < M(N) + \left[\frac{m+1}{2}\right] + 5.$$

Therefore, by the condition (10), $M(N) \ge 5$, and the property $[(m+1)/2] \le M(N)$, we obtain the desired result. $\square$

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